The global structure of spherically symmetric charged scalar field spacetimes

Jonathan Kommemi
Department of Applied Mathematics and Theoretical Physics
University of Cambridge

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Abstract

We initiate the mathematical study of spherical collapse of self-gravitating charged scalar fields. The main result gives a complete characterization of the future boundary of spacetime, providing a starting point for studying the cosmic censorship conjectures. In general, the boundary includes two null components, one emanating from the center of symmetry and the other from the future limit point of null infinity, joined by an achronal component to which the area-radius function r extends continuously to zero. Various components of the boundary, *a priori*, may be empty and establishing such generic emptiness would suffice to prove formulations of weak or strong cosmic censorship. As a simple corollary of the boundary characterization, the present paper rules out scenarios of 'naked singularity' formation by means of 'super-charging' (near-)extremal Reissner-Nordström black holes. The main difficulty in delimiting the boundary is isolated in proving a suitable global extension principle that effectively excludes a broad class of singularity formation. This suggests a new notion of 'strongly tame' matter models, which we introduce in this paper. The boundary characterization proven here extends to any such 'strongly tame' Einstein-matter system.

Contents

1 Introduction

A fundamental open problem in classical general relativity concerns the structure of singularities formed by the gravitational collapse of self-gravitating bodies. There are two conjectures of utmost consideration: weak and strong cosmic censorship. Each is a statement regarding the 'visibility' of singularities with respect to distinct notions of spacetime predictability: weak cosmic censorship is concerned with predictability in the sense of completeness of future null infinity, e.g. as given by Christodoulou [22]; strong cosmic censorship is concerned with predictability in the sense of spacetime inextendibility. The choice of terminology is a bit unfortunate; the two conjectures are, in fact, strictly speaking, logically independent.

A rigorous formulation of either conjecture is deeply rooted in the field of partial differential equations (PDEs). Indeed, the theory of general relativity itself is only described, in a mathematical context, as an initial value problem to the (second-order quasilinear hyperbolic) Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{1}$$

It is well-known that given initial data, there exists a unique maximal globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$ that solves (1) coupled to various matter models, the so-called maximal future development of initial data [10, 11, 12]. Little is known, however, about the structure of this spacetime in the large, as our understanding of global initial value problems for wave equations of such non-linearity is limited.

This paper considers the formation and characterization of singularities in spherically symmetric asymptotically flat (with one end) Einstein-Maxwell-Klein-Gordon spacetimes. Previously, these spacetimes have been studied numerically (for the collapse of a massless scalar field) by Hod and Piran [38, 39] and small data global existence has been shown by Chae [7]. This model, important to understanding many phenomena relevant to gravitational collapse, generalizes the models of Christodoulou [15] and Dafermos [25]. The self-gravitating massless real-valued scalar field model of Christodoulou is completely understood in regards to weak and strong cosmic censorship [15, 16, 21], but his model does not admit what are possibly the most relevant counter-examples to strong cosmic censorship, namely: Cauchy horizons emanating from the future limit point of null infinity. In 'coupling' an electromagnetic field to the model of Christodoulou, Dafermos is able to study the stability and instability of Cauchy horizon formation [25, 26], but his model is limited, in turn, by global topology[1] incompatible with the spacetime having only one asymptotically flat end. A further motivation for coupling

[1] The electromagnetic field is only 'coupled' to the *real-valued* scalar field via its interaction with the geometry and thus requires non-trivial topology in the initial data set for the charge to be non-zero.

an electromagnetic field, moreover, stems from the possibility of identifying charge with a 'poor man's' notion of angular momentum (cp. Reissner-Nordström and Kerr black hole solutions), providing a natural primer to the more difficult problem of non-spherical collapse. With the present model we can address both cosmic censorship conjectures within a single framework admitting many of the most fascinating features of gravitational collapse.

While we do not prove or disprove here the cosmic censorship conjectures for this model, this paper will lay a framework that will provide the necessary tools to tackle these very difficult problems in the future. The main result in Theorem 1.1 will expound on the possible global structure of spacetime, giving a characterization of its future boundary. In general, the boundary includes two null components, one emanating from the center of symmetry and the other from the future limit point of null infinity, joined by an achronal component to which the area-radius function r extends continuously to zero. Some components of the boundary may be, *a priori*, empty, and establishing such generic emptiness would suffice to prove the conjectures (see, however, §1.5.4). With respect to the state of cosmic censorship, we give an overview of results given by Christodoulou and Dafermos in §1.4. Stemming from these specific models, we are then led to give a series of conjectures in §1.5 for the general Einstein-Maxwell-Klein-Gordon system; these conjectures and their relationship to cosmic censorship are highlighted in §1.5.5. In §1.6, we announce forthcoming partial results concerning these conjectures. Within the present paper, however, we bolster the case for the validity of weak cosmic censorship by immediately ruling out (as a corollary of Theorem 1.1) the possibility, entertained in the physics literature, e.g. [4, 9, 34, 42, 43, 52, 61, 62, 66], of creating 'naked singularities' by 'super-charging' (near-)extremal Reissner-Nordström black holes.

The main difficulty in the proof of Theorem 1.1 is establishing a stronger characterization of 'first singularities' than that proposed by Dafermos in [28]. We will call this stronger characterization the 'generalized extension principle' to distinguish it from (what we shall call) the 'weak extension principle' of [28]. The 'generalized extension principle' is formulated very generally without reference to the topology or the geometry of the spherically symmetric initial data (so as to be applicable as well in the cosmological setting or in the case with two asymptotically flat ends). See §1.8.1 for a formal definition. In §4.3, we will show that Einstein-Maxwell-Klein-Gordon satisfies both extension principles, specifically: A 'first singularity' must emanate from a spacetime boundary to which the area-radius function r extends continuously to zero.

Given the 'generalized extension principle', the proof of Theorem 1.1 follows from monotonicity arguments derived from the dominant energy condition[2] and is independent of any other structure of the system. This observation will motivate a notion of 'strongly tame' and 'weakly tame' Einstein-matter systems introduced in this paper. In the case of a 'strongly tame' system, i.e. one which satisfies the 'generalized extension principle' and obeys the dominant energy condition, the conclusion of Theorem 1.1 holds. In the case of a 'weakly tame' system, i.e. one which satisfies the 'weak extension principle' and obeys the dominant energy condition, parts of the conclusion of Theorem 1.1 still hold, in particular, those most relevant for the study of weak cosmic censorship but not strong cosmic censorship. See §1.8.2–1.8.4 for a discussion.

We note finally that many of the conjectures in §1.5 are not model-specific and rely simply on parts of the conclusion of Theorem 1.1; these conjectures, in turn, can be read more generally so as to apply to 'strongly tame' and 'weakly tame' systems (where appropriate). This paper, as such, provides a blueprint for establishing the global structure of spherically symmetric spacetimes.

[2]Much of Theorem 1.1, in fact, uses the monotonicity governed by Raychaudhuri's equation, which just needs the null energy condition (cf. the proof of Theorem 1.1 in §5).

1.1 Self-gravitating charged scalar field model

We briefly summarize the mathematical framework necessary to describe the self-gravitating charged scalar field model.

1.1.1 Principal $U(1)$-bundles and associated complex line bundles

Consider a 4-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ and a principal $U(1)$-bundle $\pi : P \to \mathcal{M}$. Let ω be a $\mathfrak{u}(1)$-valued connection defined on P and let $\mathcal{F} = \mathrm{d}\omega$ denote its curvature.

Given a local section $s : U \subset \mathcal{M} \to \pi^{-1}(U)$ of the principal bundle (called a gauge choice), we define a local gauge potential on $\mathcal{M}$ as the $\mathfrak{u}(1)$-valued 1-form $\tilde{A} = s^*\omega$ and its (electromagnetic) field strength as the $\mathfrak{u}(1)$-valued 2-form $\tilde{F} = s^*\mathcal{F} = \mathrm{d}\tilde{A}$.

Let $f : P \to P$ be a diffeomorphism of P that commutes with the $U(1)$-action, i.e. $f(p{\cdot}g) = f(p) \cdot g$, and fixes fibers of P, i.e. $\pi \circ f = \pi$, (called a gauge transformation). We then define a mapping $f_g : P \to U(1)$ by $f(p) = p \cdot f_g(p)$ such that $f_g(p \cdot g) = g^{-1} \cdot f_g(p) \cdot g$. The local gauge potential and its local field strength transform (on each fiber) as

$$\begin{aligned} \tilde{A}_g = s^*\left(f^*\omega(p)\right) &= f_g(p)^{-1} \cdot \tilde{A} \cdot f_g(p) + f_g(p)^{-1}\mathrm{d}f_g(p) \\ \tilde{F}_g = s^*\left(f^*\mathcal{F}(p)\right) &= f_g(p)^{-1} \cdot \tilde{F} \cdot f_g(p). \end{aligned}$$

Since $U(1)$ is abelian, $\tilde{F}$ is invariant under gauge transformations, i.e.

$$\tilde{F}_g = \tilde{F}.$$

Identifying the Lie algebra $\mathfrak{u}(1)$ with $\mathrm{i}\mathbb{R}$, we can define a (local) $\mathbb{R}$-valued 1-form A on $\mathcal{M}$ such that

$$\tilde{A} = \mathrm{i}A$$

and hence define a (global) $\mathbb{R}$-valued 2-form F on $\mathcal{M}$ such that

$$F = \mathrm{d}A.$$

Fix $\mathfrak{e} \in \mathbb{Z}$. Let $\varrho : U(1) \to \mathbb{C}$ be a representation of $U(1)$ on the vector space $\mathbb{C}$ (over the field $\mathbb{C}$) given by $\varrho(g) = g^{\mathfrak{e}}$. Define

$$P \times_\varrho \mathbb{C} = (P \times \mathbb{C}) \,/ \sim$$

where $(p_1, z_1) \sim (p_2, z_2)$ if there exists $g \in U(1)$ such that $p_2 = p_1 \cdot g$ and $z_2 = \varrho(g^{-1}) \cdot z_1$. We call the fiber bundle

$$\pi_\varrho : P \times_\varrho \mathbb{C} \to \mathcal{M}, \qquad [p, z] \mapsto \pi(p)$$

the associated complex line bundle through the representation ϱ.

A charged scalar field ϕ is a global section of $P \times_\varrho \mathbb{C}$ and corresponds to an equivariant $\mathbb{C}$-valued map on P, i.e.

$$\phi(p \cdot g) = \varrho(g^{-1}) \cdot \phi(p)$$

for all $p \in P$ and all $g \in U(1)$.

Let ω_ϱ be the associated connection on $P \times_\varrho \mathbb{C}$. This connection is defined so that, locally on $\mathcal{M}$, (suppressing the pullback of the local section)

$$\tilde{A}_\varrho = \varrho_* \tilde{A}.$$

The exterior covariant derivative (called the gauge derivative) on sections of $P \times_\varrho \mathbb{C}$ is then defined[3] so that, locally on $\mathcal{M}$,

$$\mathrm{D} = \mathrm{d} + \tilde{A}_\varrho = \mathrm{d} + \mathfrak{e}\mathrm{i}A.$$

If $\mathfrak{e} = 0$, then the structure group reduces to $SO(1)$ and ϕ is just a global section of (the trivial associated bundle) $\mathcal{M} \times \mathbb{R}$, i.e. ϕ is a $\mathbb{R}$-valued map on $\mathcal{M}$.

[3] The definition of the exterior covariant derivative is independent of the metric on $\mathcal{M}$, but transforms 'tensorially' under a gauge transformation. Equivalently, we can write $\mathrm{D} = \nabla + \mathfrak{e}\mathrm{i}A$.

1.1.2 System of equations

The charged scalar field model is described by the collection (as defined in §1.1.1)

$$\{(\mathcal{M}, g_{\mu\nu}),\ \pi : P \to \mathcal{M},\ \omega,\ \pi_\varrho : P \times_\varrho \mathbb{C},\ \phi,\ \mathfrak{e}\}$$

satisfying the Einstein-Maxwell-Klein-Gordon equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} = 8\pi \left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{KG}}_{\mu\nu}\right) \tag{2}$$

$$T^{\mathrm{EM}}_{\mu\nu} = \frac{1}{4\pi} \left(g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \tag{3}$$

$$T^{\mathrm{KG}}_{\mu\nu} = \frac{1}{2} \mathrm{D}_\mu \phi \left(\mathrm{D}_\nu \phi\right)^\dagger + \frac{1}{2} \mathrm{D}_\nu \phi \left(\mathrm{D}_\mu \phi\right)^\dagger - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \mathrm{D}_\alpha \phi \left(\mathrm{D}_\beta \phi\right)^\dagger + \mathfrak{m}^2 \phi \phi^\dagger \right) \tag{4}$$

$$\nabla^\nu F_{\mu\nu} = \mathfrak{e} i 4\pi \left(\phi \left(\mathrm{D}_\mu \phi\right)^\dagger - \phi^\dagger \mathrm{D}_\mu \phi \right) \tag{5}$$

$$g^{\mu\nu} \mathrm{D}_\mu \mathrm{D}_\nu \phi = \mathfrak{m}^2 \phi, \tag{6}$$

where the coupling constant $\mathfrak{e}$ is to be interpreted as the (electromagnetic) charge of the scalar field having mass $\mathfrak{m}^2$.

1.1.3 Dominant energy condition

To ensure that the matter model obeys the dominant energy condition, we must require that $\mathfrak{m}^2 \geq 0$ (cf. §2.2.2).

We note that, however, the 'generalized extension principle' for the Einstein-Klein-Gordon system ($\mathfrak{e} = F_{\mu\nu} = 0$) can be established for arbitrary $\mathfrak{m}^2$ (cf. footnote 50).

1.1.4 Well-posedness

An easy generalization of a theorem of Choquét-Bruhat and Geroch [12] gives a unique smooth maximal future development $(\mathcal{M}, g_{\mu\nu}, \phi, F_{\mu\nu})$ satisfying (2)–(6) for given smooth initial data defined on a Cauchy surface $\Sigma^{(3)}$ (in our convention, $\mathcal{M}$ is a manifold-with-boundary with (past) boundary $\Sigma^{(3)}$). For the purpose of this paper, it will not be necessary to concern ourselves with a general discussion of the constraint equations and the construction of such initial data sets, as we shall restrict to spherical symmetry where the relevant considerations are straightforward. See §2.1.

1.2 Theorem 1.1: global characterization of spacetime

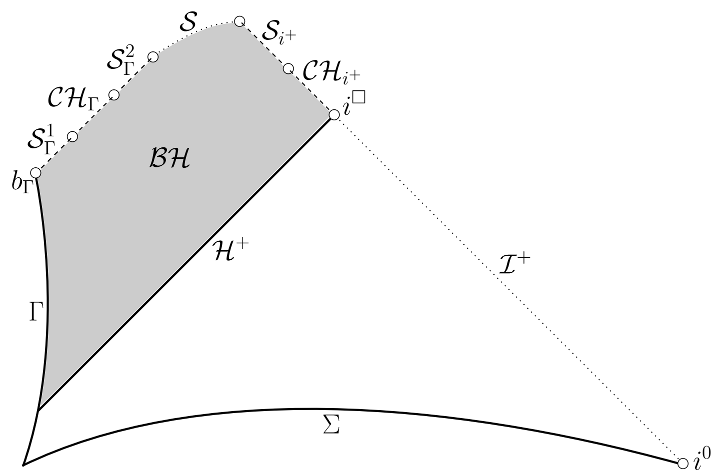
The main result (Theorem 1.1) of this paper concerns the general possible structure of a spherically symmetric Einstein-Maxwell-Klein-Gordon spacetime arising from gravitational collapse. This global characterization already captures non-trivial aspects of the dynamics of (2)–(6) and can be considered a starting point for further study of the cosmic censorship conjectures, whose statements are given in §1.3.

As the proof of Theorem 1.1 will make clear, this characterization holds for a larger class of spherically symmetric Einstein-matter systems, which we call 'strongly tame'. In fact, much of Theorem 1.1 will hold for a still larger class of 'weakly tame' Einstein-matter systems. With this in mind, in the statement of Theorem 1.1 below, a box will indicate the specific appeal to an Einstein-matter system being 'strongly tame', otherwise the model need only be 'weakly tame' for the assertion to hold. See §1.8.2–1.8.4 for a discussion and the statements of Theorems 1.12 and 1.13.

Theorem 1.1. *Let* $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ *denote the maximal future development of smooth spherically symmetric asymptotically flat with one end initial data for the Einstein-Maxwell-Klein-Gordon system containing no anti-trapped*[4] *spheres of symmetry, where* $r : \mathcal{Q}^+ \to [0, \infty)$ *is the area-radius function.*

I: Penrose diagram

The Penrose diagram[5] *of* $\mathcal{Q}^+$ *is as depicted*



with boundary $\Sigma \cup \Gamma$ *(in the sense of manifold-with-boundary) and boundary* $\mathcal{B}^+$ *induced by the the ambient manifold* $\mathbb{R}^{1+1}$ *admitting a decomposition*

$$\mathcal{B}^+ = b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma \cup \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square \cup \mathcal{I}^+ \cup i^0 \tag{7}$$

to be enumerated immediately below.

II: Boundary characterization

The spacetime boundary is described as follows:

Boundary in the sense of manifold-with-boundary

Σ *is the spacelike past boundary of* $\mathcal{Q}^+$ *and is the projection to* $\mathcal{Q}^+$ *of the initial Cauchy hypersurface* $\Sigma^{(3)}$ *in* $\mathcal{M}$.

Γ *is the timelike boundary of* $\mathcal{Q}^+$ *on which* $r = 0$ *and is the projection to* $\mathcal{Q}^+$ *of the set of fixed points of the group action* $SO(3)$ *on* $\mathcal{M}$.

[4] A point $p \in \mathcal{Q}^+$ is an anti-trapped sphere of symmetry if the ingoing null derivative of r, evaluated at p, is non-negative. (This is what sometimes would be called '*past outer* trapped or marginally trapped'.)

[5] A Penrose diagram is the range of globally-defined bounded double null co-ordinates as a subset of $\mathbb{R}^{1+1}$ (cf. §2.1). For spherically symmetric spacetimes, the diagrams conveniently help convey global causal-geometric information about the metric. Readers unfamiliar with Penrose diagrams should consult the appendix of [33].

Boundary induced from the ambient $\mathbb{R}^{1+1}$

i^0 is the unique limit point of Σ in $\overline{\mathcal{Q}^+}\setminus\mathcal{Q}^+$.[6] *$r$ extends continuously*[7] *to ∞ on i^0.*

$\mathcal{I}^+$ is a connected non-empty open null segment emanating from (but not including) i^0 characterized by the set of $p \in \overline{\mathcal{Q}^+}\setminus\mathcal{Q}^+$ that are limit points of outgoing null rays in $\mathcal{Q}^+$ for which $r \to \infty$. r extends continuously to ∞ on $\mathcal{I}^+$.

$i^\square$ is the unique future limit point of $\mathcal{I}^+$.

$\mathcal{CH}_{i^+}$ is a connected (possibly empty) half-open null segment emanating[8] *from (but not including) $i^\square$. r extends*[9] *to a function on $\mathcal{CH}_{i^+}$ that is non-zero except possibly at its future endpoint.*

$\mathcal{S}_{i^+}$ is a connected (possibly empty) half-open null segment emanating from (but not including) the future endpoint of $\mathcal{CH}_{i^+} \cup i^\square$. r extends continuously to zero on $\mathcal{S}_{i^+}$.

b_Γ is the unique future limit point of Γ in $\overline{\mathcal{Q}^+}\setminus\mathcal{Q}^+$. r extends continuously to zero on $b_\Gamma \setminus \left(\mathcal{CH}_{i^+} \cup i^\square\right)$.

$\mathcal{S}^1_\Gamma$ is a connected (possibly empty) half-open null segment emanating from (but not including) b_Γ. r extends continuously to zero on $\mathcal{S}^1_\Gamma \setminus \left(\mathcal{CH}_{i^+} \cup i^\square\right)$.

$\mathcal{CH}_\Gamma$ is a connected (possibly empty) half-open null segment emanating from (but not including) the future endpoint of $b_\Gamma \cup \mathcal{S}^1_\Gamma$. r extends continuously to a non-zero function on $\mathcal{CH}_\Gamma$ except possibly at its future endpoint.

$\mathcal{S}^2_\Gamma$ is a connected (possibly empty) half-open null segment emanating from (but not including) the future endpoint of $\mathcal{CH}_\Gamma$. r extends continuously to zero on $\mathcal{S}^2_\Gamma$.

$\mathcal{S}$ is a connected (possibly empty) achronal curve that does not intersect the null rays emanating from limit points b_Γ and $i^\square$. $\boxed{r \text{ extends continuously to zero on } \mathcal{S}.}$

Common intersection of the boundary components

Σ and Γ intersect at a single point.

If $\mathcal{S} = \emptyset$, the future endpoint of $b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma \cup \mathcal{S}^2_\Gamma$ coincides with the future endpoint of $\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square$.

Modulo these common intersections, the boundary decomposition (7) is disjoint.

III: Completeness of $\mathcal{I}^+$

If either of the following hold:

1. *$\mathcal{BH} = \mathcal{Q}^+ \setminus J^-(\mathcal{I}^+) \neq \emptyset$; or,*
2. *$\sup_{\mathcal{CH}_\Gamma} r < \infty$,*[10]

then $\mathcal{I}^+$ is complete in the sense of Christodoulou [22].

[6] The closure is with respect to a bounded conformal representation of $\mathcal{Q}^+$ into the ambient manifold $\mathbb{R}^{1+1}$. See §2.1. Similarly, causal-geometric constructions, e.g. the causal future J^+, the chronological past I^-, etc., will be with respect to the topology and the causal structure of the ambient $\mathbb{R}^{1+1}$.

[7] By this we mean here, and in what follows: r extends continuously to a $[0,\infty]$-valued function on $\mathcal{Q}^+ \cup i^0$ so as to yield ∞ on i^0.

[8] The 'Cauchy horizon' $\mathcal{CH}_{i^+}$ will have special significance within the context of this paper, for it will be the only type of Cauchy horizon that is non-'first singularity'-emanating (NFSE). See §1.7.

[9] The extension, which need not be continuous (see, however, Statement IV.4 below), is given by monotonicity along outgoing null curves.

[10] If $\mathcal{CH}_\Gamma = \emptyset$, then condition 2 is trivially satisfied, as we take the convention $\sup \emptyset = -\infty$.

[The completeness condition of [22], in the present context, takes the following form: Consider the parallel transport of an ingoing null vector X along a fixed outgoing null segment in $\mathcal{Q}^+$ that has a limit point on $\mathcal{I}^+$. The affine length of integral curves of X (in fact, restricted to $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$) tends to ∞ as $\mathcal{I}^+$ is approached.]

If $\mathcal{I}^+$ is complete, we write[11]

$$i^\square = i^+.$$

Alternatively, when $\mathcal{I}^+$ is not complete, we write

$$i^\square = i^{\text{naked}}$$

and we say that $(\mathcal{M}, g_{\mu\nu})$ is a 'naked singularity' spacetime.

IV: Geometry of the trapped region

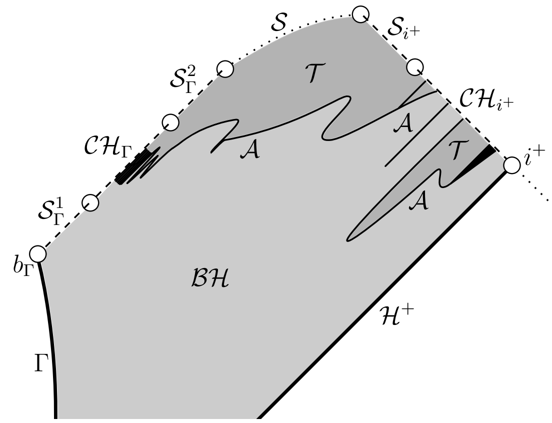
Let $\mathcal{R}$ be the non-empty 'regular region' defined as the set of all $p \in \mathcal{Q}^+$ for which the outgoing null derivative of r is positive. Let $\mathcal{A}$ be the (possibly empty) 'apparent horizon' defined as the set of all $p \in \mathcal{Q}^+$ for which the outgoing null derivative of r vanishes. Let $\mathcal{T}$ be the (possibly empty) 'trapped region' defined as the set of all $p \in \mathcal{Q}^+$ for which the outgoing null derivative of r is negative.[12]

1. $\Gamma \subset \mathcal{R}$.
2. *Consider $p, p' \in \mathcal{Q}^+$ along an outgoing null ray with p' to the future of p.*
   a. *If $p \in \mathcal{A}$, then $p' \in \mathcal{A} \cup \mathcal{T}$.*
   b. *If $p \in \mathcal{T}$, then $p' \in \mathcal{T}$.*

   In particular, $(\mathcal{A} \cup \mathcal{T}) \cap J^-(\mathcal{I}^+) = \emptyset$.
3. $\boxed{\text{If } \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+} \neq \emptyset, \text{ then } \mathcal{A} \cup \mathcal{T} \neq \emptyset.}$ *(If $\mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+} = \emptyset$, then $\mathcal{A} \cup \mathcal{T}$ is possibly empty.)*
4. *Let $p \in \mathcal{CH}_{i^+}$ that is not the future endpoint of $\mathcal{CH}_{i^+}$. If there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of p such that either $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A}$ or $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{T}$, then r extends continuously on $(\mathcal{U} \cap \mathcal{Q}^+) \cup \mathcal{CH}_{i^+}$.*
5. *The apparent horizon $\mathcal{A}$ is clearly a closed set in $\mathcal{Q}^+$. Consider, however, the limit points of $\mathcal{A}$ on the boundary $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ in the topology of $\mathbb{R}^{1+1}$.*
   a. $\boxed{\text{If } \mathcal{A} \neq \emptyset, \text{ then all limit points of } \mathcal{A} \text{ that lie on the boundary } \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+ \text{ lie on } \mathcal{CH}_{i^+} \cup i^+ \text{ and on a (possibly degenerate) closed, necessarily connected interval of } b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma.}$
   b. *If $\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ has a limit point on $\mathcal{CH}_{i^+} \cup i^+$.*
   c. *If $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a limit point on $\mathcal{CH}_\Gamma$, then there are no limit points of $\mathcal{A}$ on $b_\Gamma \cup \mathcal{S}^1_\Gamma$.*
   d. *If $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a limit point on $b_\Gamma \cup \mathcal{S}^1_\Gamma$, then $\mathcal{CH}_\Gamma = \emptyset$.*
   e. $\boxed{\text{If } \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+} \neq \emptyset, \text{ then } \mathcal{A} \text{ has a limit point on } b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma.}$

[11] This explains the choice of notation $\mathcal{S}_{i^+}$ and $\mathcal{CH}_{i^+}$: If either of these sets are non-empty, then $i^\square = i^+$, since condition 1 is satisfied.

[12] The 'apparent horizon' is thus the set of symmetry spheres that are marginally trapped; the 'trapped region' is the set of symmetry spheres that are trapped.



V: Properties of the Hawking mass

Let m be the Hawking mass function given by

$$1 - \frac{2m}{r} = g(\nabla r, \nabla r).$$

1. *If $p \in \mathcal{R} \cup \mathcal{A}$, then $m(p)$ is non-decreasing in the future-directed outgoing null direction and is non-increasing in the future-directed ingoing null direction. In particular, m extends to a non-increasing, non-negative function along $\mathcal{I}^+$ and we define the final Bondi mass M_f of the spacetime by*

$$M_f = \inf_{\mathcal{I}^+} m.$$

2. *The following relations hold:*

$$\begin{array}{ccccc} \frac{2m}{r} & < & 1 & \text{in} & \mathcal{R} \\ \frac{2m}{r} & = & 1 & \text{in} & \mathcal{A} \\ \frac{2m}{r} & > & 1 & \text{in} & \mathcal{T}. \end{array}$$

VI: Penrose inequality

Let $\mathcal{H}^+ \subset \mathcal{R} \cup \mathcal{A}$ denote the (possibly empty) half-open outgoing null segment forming the past-boundary of the (possibly empty) black hole region $\mathcal{BH}$, i.e.

$$\mathcal{H}^+ = \left(\overline{J^-(\mathcal{I}^+)} \cap \mathcal{Q}^+\right) \setminus J^-(\mathcal{I}^+).$$

We note that if $\mathcal{H}^+ \neq \emptyset$, then $\mathcal{H}^+$ has a past endpoint on $\Sigma \cup \Gamma$.[13]

If $\mathcal{BH} \neq \emptyset$, then $\mathcal{H}^+ \neq \emptyset$ and, moreover, the following inequality holds:

$$\sup_{\mathcal{H}^+} r \leq 2M_f.$$

In particular, if $\mathcal{BH} \neq \emptyset$, then $M_f > 0$.

[13] In the above diagram, we have depicted $\mathcal{H}^+$ such that $\mathcal{H}^+ \cap \Sigma = \emptyset$, but, indeed, it may be that $\mathcal{H}^+ \cap \Sigma \neq \emptyset$ (cf. Theorem 1.8 in §1.6.1).

VII: Extendibility of the solution

1. *The Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is a continuous $[0,\infty]$-valued function on $\mathcal{Q}^+ \cup \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}$ that yields ∞ on $\mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}$.*

2. *Let $p \in \mathcal{S}^1_\Gamma$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of p.*

   a. *There exists a sequence $\{p_j\}_{j=1}^\infty \subset \mathcal{U} \cap \mathcal{Q}^+$ with $p_j \to p$ such that*

   $$\limsup_{j\to\infty} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}(p_j) = \infty.$$

   b. *If $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A} \cup \mathcal{T}$, then the Kretschmann scalar is a continuous $[0,\infty]$-valued function on $(\mathcal{U} \cap \mathcal{Q}^+) \cup \mathcal{S}^1_\Gamma$ that yields ∞ on $\mathcal{S}^1_\Gamma \cap \mathcal{U}$.*

3. *If $\mathcal{CH}_{i^+} = \emptyset$, then $\mathcal{H}^+$ is affine complete.*

4. *If $(\mathcal{M}, g_{\mu\nu})$ is future-extendible as a C^2-Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}})$, then there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting the manifold $\mathcal{M}$ such that the closure of the projection of $\gamma|_\mathcal{M}$ to $\mathcal{Q}^+$ intersects $\mathcal{CH}_\Gamma \cup \mathcal{CH}_{i^+}$. In particular, if $\mathcal{CH}_\Gamma \cup \mathcal{CH}_{i^+} = \emptyset$, then $(\mathcal{M}, g_{\mu\nu})$ is C^2-future-inextendible.*

1.3 Weak and strong cosmic censorship

We will discuss known (limited) results concerning cosmic censorship for the Einstein-Maxwell-Klein-Gordon system in §1.4 and establish more conjectures in §1.5 that will imply, in particular, cosmic censorship. To aid this study, it will be convenient to include here concise statements of the cosmic censorship conjectures (following Christodoulou in [22]) under the framework presented by Theorem 1.1.

Conjecture 1.1 (**Weak cosmic censorship**)**.** *For generic initial data as in Theorem 1.1, $\mathcal{I}^+$ is complete.*

Conjecture 1.2 (**Strong cosmic censorship**)**.** *For generic initial data as in Theorem 1.1, the maximal future development is future-inextendible as a sufficiently smooth Lorentzian metric.*

The question of regularity in Conjecture 1.2 will be discussed in §1.4.2, §1.4.4, §1.5.4 and §1.6.2.

1.4 Models of Christodoulou and Dafermos

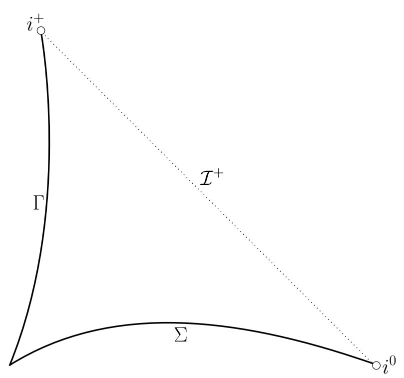
Contained within Theorem 1.1 is the self-gravitating massless real-valued scalar field model of Christodoulou [15]; this corresponds to taking $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$. The model of Dafermos [25], i.e the model for which $\mathfrak{m}^2 = \mathfrak{e} = 0$, but $F_{\mu\nu}$ is not assumed to vanish, is not, however, included in the statement of Theorem 1.1 in view of the topology of the initial data. If we impose that Σ has one asymptotically flat end and $\mathfrak{m}^2 = \mathfrak{e} = 0$, then the model of Dafermos, necessarily, reduces to that of Christodoulou, i.e. it follows that $F_{\mu\nu} = 0$.

1.4.1 Christodoulou: the uncharged massless case

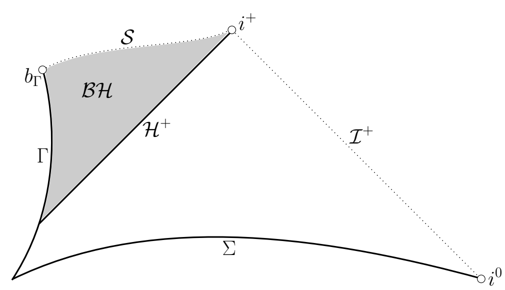
First some preliminaries. In the case $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$, the system (2)–(6) exhibits stronger monotonicity properties not present in the more general case. In particular, we can strengthen the boundary characterization of Theorem 1.1 as follows:

1. $\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} = \emptyset$.
2. $\mathcal{S}^2_\Gamma = \emptyset$, and $\mathcal{S}_\Gamma := \mathcal{S}^1_\Gamma$.
3. $\mathcal{S}$ is C^1-spacelike.
4. If $\mathcal{S} \neq \emptyset$, then either
    a. $\mathcal{S}_\Gamma \cup \mathcal{CH}_\Gamma = \emptyset$; or,
    b. $\mathcal{CH}_\Gamma = \emptyset$ and $\mathcal{S}_\Gamma \neq \emptyset$; or,
    c. $\mathcal{CH}_\Gamma \neq \emptyset$ and $\mathcal{S}_\Gamma = \emptyset$.
5. If $\mathcal{S} = \emptyset$, then either
    a. $\mathcal{S}_\Gamma \cup \mathcal{CH}_\Gamma = \emptyset$; or
    b. $\mathcal{CH}_\Gamma = \emptyset$ and $\mathcal{S}_\Gamma \neq \emptyset$; or,
    c. $\mathcal{CH}_\Gamma \neq \emptyset$ and $\mathcal{S}_\Gamma = \emptyset$.

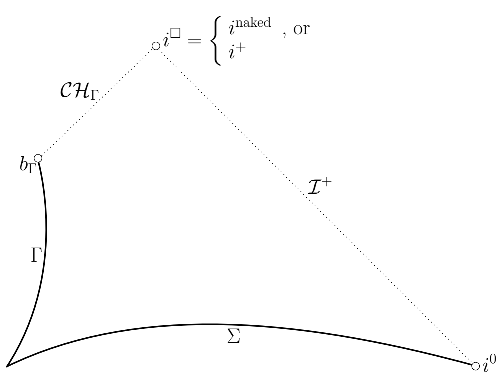
In all, there are six possible spacetimes as depicted below in diagrams I–VI.[14]



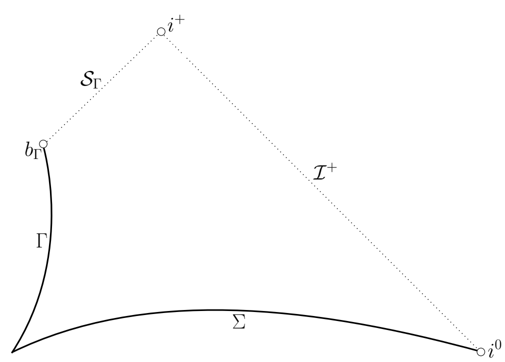
I: dispersive



II: black hole with only a spacelike singularity

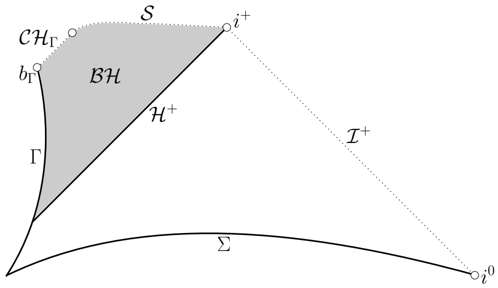


III: 'light cone singularity'

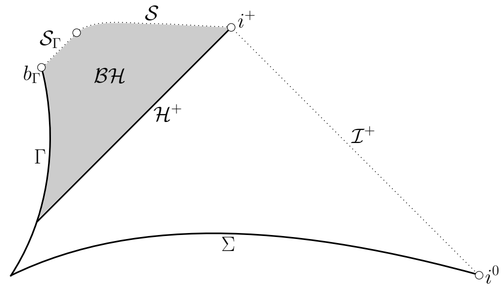


IV: 'collapsed light cone singularity'

[14] We do not differentiate between the cases in which the past endpoint of $\mathcal{H}^+$ intersects $\Gamma \backslash \Sigma$, $\Sigma \backslash \Gamma$ or $\Sigma \cap \Gamma$. This is related, however, to the important issue of dynamic formation of black holes considered in §1.6.1.



V: black hole with central Cauchy horizon



VI: black hole with 'collapsed light cone'

'Light cone singularity' spacetimes, as in diagram III above, *a priori*, may, or may not, have a complete future null infinity $\mathcal{I}^+$ and hence, may, or may not, have (in our convention) a 'future timelike infinity' i^+. Moreover, such spacetimes may, or may not, be future-extendible beyond $\mathcal{CH}_\Gamma$. In principle, there may exist, in particular, a spacetime as in diagram III where $i^\square = i^{\text{naked}}$, but for which the solution is future-inextendible. This illustrates why strong cosmic censorship does *not* imply weak cosmic censorship.

In [16, 17], Christodoulou shows that, in fact, all these spacetimes (I–VI) as given by Theorem 1.1 occur, constructing, in particular, examples in which $\mathcal{I}^+$ is incomplete (diagram III: 'light cone singularity' with $i^\square = i^{\text{naked}}$, which we, henceforth, call a 'naked singularity') but the spacetime is also future-extendible. This demonstrates the necessity of having a genericity assumption in the formulation of Conjectures 1.1 and 1.2. We note that the solutions (I–VI) constructed are not smooth but, nonetheless, lie in a 'BV' class for which strong well-posedness can still be proven (see the discussion in §1.8.7). They are thus, in every sense, strong solutions.

In his seminal work [21], however, Christodoulou shows that the set of solutions with Penrose diagram III–VI are non-generic, more precisely, these solutions form a set of positive co-dimension in the family of all solutions as above. This is summarized in

Theorem 1.2 (Christodoulou)**.** *For all initial data as in Theorem 1.1 in the more general BV class with $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$, the maximal future development has $\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} = \emptyset$. Moreover, for generic initial data, $\mathcal{S}_\Gamma \cup \mathcal{CH}_\Gamma = \emptyset$ and the spacetimes are inextendible as C^0-Lorentzian metrics. In particular, Conjectures 1.1 and 1.2 are* true *in the case of a self-gravitating massless real-valued scalar field, and generic spacetimes are as depicted in diagrams I and II above.*[15]

For a discussion of the significance of the positive resolution of cosmic censorship for the massless real-valued scalar field in the context of other models, in particular Einstein-dust, see §1.8.8.

In proving Theorem 1.2, Christodoulou makes use of the following result, which is also of independent interest.

[15] With respect to Conjecture 1.2, C^2-inextendibility of the spacetime would follow by Statement VII of Theorem 1.1, but the regularity class considered by Christodoulou in [16, 17] is below C^2 for the metric, hence the desirability of the stronger C^0-formulation, which Christodoulou indeed obtains by a separate argument. For a general Einstein-Maxwell-Klein-Gordon spacetime, however, it is conjectured (v. Conjecture 1.9) that such a strong formulation of Conjecture 1.2 will not hold (cf. Theorems 1.4 and 1.9).

Theorem 1.3 (Christodoulou)**.** *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ be the development of initial data as in Theorem 1.1 with $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$. For $p, p' \in \mathcal{R}$ along an outgoing null ray C_0^+, with p' to the future of p, suppose the ingoing null ray C_p^- that emanates from p terminates on $q \in \Gamma \cup b_\Gamma$. Let δ_0 and η_0 be defined by*[16]

$$\delta_0 = \frac{r(p')}{r(p)} - 1 \qquad \textit{and} \qquad \eta_0 = \frac{2\,(m(p') - m(p))}{r(p')},$$

where m is the Hawking mass function.

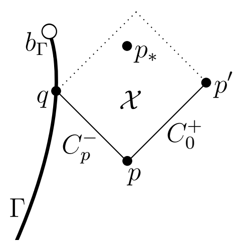
There are positive constants c_1 and c_2 such that if $\delta_0 \le c_1$ and

$$\eta_0 > c_2 \delta_0 \log\left(\delta_0^{-1}\right),$$

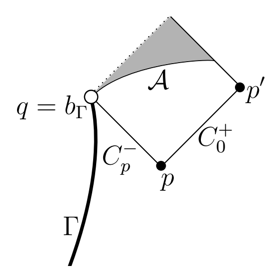
then the region $\mathcal{X} \subset \mathcal{Q}^+$ given by

$$\mathcal{X} = \left(J^+(p) \cap \mathcal{Q}^+\right) \setminus \left(J^+(q) \cup J^+(p')\right)$$

contains a trapped surface $p_* \in \mathcal{X}$ as depicted below.



Christodoulou applies Theorem 1.3 as an auxiliary lemma in the context of the proof of Theorem 1.2. One begins with a spacetime as given by diagrams III–VI, and the goal is to produce a 1-parameter family of spacetimes containing the given one such that *all* other members of the family have $\mathcal{A} \neq \emptyset$ with limit point $b_\Gamma \neq i^+$. The infinite blue-shift along C_p^- plays an important role in the proof of Theorem 1.2, for it provides the linear mechanism for instability.[17] Using this effect, it is shown that for the perturbed spacetimes, the assumptions of Theorem 1.3 hold with $q = b_\Gamma \neq i^+$ and a sequence of $p, p' \to q$.



Thus, Theorem 1.3 applies to yield $\mathcal{A}$ as desired. It is interesting that in Christodoulou's construction the 1-parameter family of perturbations coincide with the original spacetime in the past of $q = b_\Gamma$.

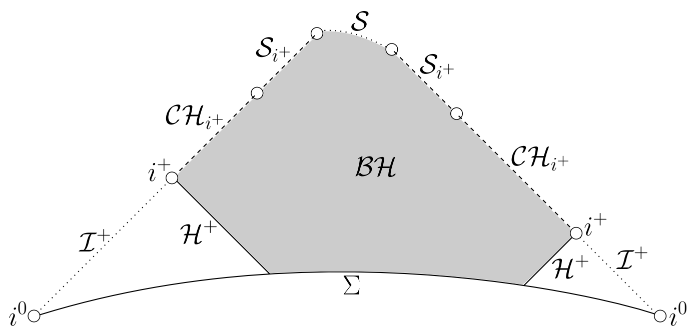
[16] The constants δ_0 and η_0 give the dimensionless size and the dimensionless mass content, respectively, of the enclosed annular region bounded by p and p'.

[17] Once this property of $\mathcal{A}$ is established, the emptiness of $\mathcal{S}_\Gamma \cup \mathcal{CH}_\Gamma$ is a consequence of the special monotonicity $\partial_u \partial_v r < 0$ in the trapped region.

1.4.2 Dafermos: the massless case with topological charge

Dafermos considers the model for which $\mathfrak{m}^2 = \mathfrak{e} = 0$ and $F_{\mu\nu} \neq 0$. Since the scalar field is itself uncharged, $F_{\mu\nu}$ can be non-trivial only if the Cauchy surface Σ has two asymptotically flat ends. In this case, however, the electromagnetic field is only 'coupled' to the scalar field via its interaction with the geometry.

An analogue of Theorem 1.1, applied to this class of initial data, yields a Penrose diagram as depicted below.



Black hole with two asymptotically flat ends

One easily infers that for a spacetime having two asymptotically flat ends, the black hole region $\mathcal{BH}$ is necessarily non-empty and, therefore, by the analogue[18] of Theorem 1.1, both connected components of $\mathcal{I}^+$ are complete. Thus, weak cosmic censorship is trivially true but not very physically interesting. On the other hand, this model is well-suited for addressing strong cosmic censorship in a non-trivial context because it admits as a special solution the Reissner-Nordström family, with parameters M_+ and Q_+, where, if $0 < |Q_+| < M_+$, then $\mathcal{CH}_{i^+}$ is non-empty and the maximal future development is future-extendible as a smooth Lorentzian metric. Thus, for strong cosmic censorship to be true, the Reissner-Nordström solution, in particular, must be shown to be 'unstable'.

In considering this issue of stability, Dafermos shows, however, that whenever a black hole is 'sub-extremal in the limit' and $Q_+ \neq 0$, then $\mathcal{CH}_{i^+}$ is non-empty and the maximal future development is continuously extendible [26]. Indeed, these assumptions can be shown to hold for solutions arising from arbitrary data in a suitable 'open neighborhood' of Reissner-Nordström initial data; in particular, the spherically symmetric C^0-formulation[19] of strong cosmic censorship is *false*! Before presenting this result, it will be convenient to discuss asymptotic parameters of black hole solutions, i.e. solutions with $\mathcal{BH} \neq \emptyset$, arising when $\mathfrak{m}^2 = \mathfrak{e} = 0$, namely: area, mass and charge.

The asymptotic area r_+ of the black hole (as measured along $\mathcal{H}^+$), given by

$$r_+ = \sup_{\mathcal{H}^+} r,$$

is well-defined by monotonicity and is finite by Statement VI of the analogue of Theorem 1.1. Similarly by monotonicity, the asymptotic mass m_+ of the black hole (as measured along $\mathcal{H}^+$), given by

$$m_+ = \sup_{\mathcal{H}^+} m,$$

where m is the Hawking mass function, is well-defined and finite.[20]

[18]In this model note that there are, in general, anti-trapped regions. To prove the analogue of Theorem 1.1, it suffices to assume that there exists a point $(u', v') \in \Sigma$ such that $\partial_v r < 0$ in $\Sigma \cap \{v \leq v'\}$ and $\partial_u r < 0$ in $\Sigma \cap \{v \geq v'\}$.

[19]where in the analogue of Conjecture 1.2, 'sufficiently smooth' means 'continuous'

[20]Indeed, since $\mathcal{H}^+ \subset \mathcal{R} \cup \mathcal{A}$, one has $m_+ \leq \frac{1}{2} r_+$.

In the case $\mathfrak{e} = 0$, the scalar invariant $F_{\mu\nu}F^{\mu\nu}$ is given by

$$F_{\mu\nu}F^{\mu\nu} = -\frac{8\pi}{r^4}\left(Q_{\mathrm{e}}^2 - Q_{\mathrm{m}}^2\right)$$

for constants $Q_{\mathrm{e}}, Q_{\mathrm{m}} \in \mathbb{R}$.[21] The constant Q_+ such that $Q_+^2 = Q_{\mathrm{e}}^2 + Q_{\mathrm{m}}^2$, defines, in particular, the asymptotic charge[22] of the black hole.

For convenience, we also define the asymptotic re-normalized mass ϖ_+ by

$$\varpi_+ = \sup_{\mathcal{H}^+} \varpi := \sup_{\mathcal{H}^+} \left(m + \frac{2\pi Q_+^2}{r}\right) = m_+ + \frac{2\pi Q_+^2}{r_+}.$$

In the case of Reissner-Nordström, $\varpi = \varpi_+ = M_+$.

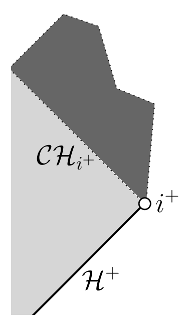
We now state

Theorem 1.4 (Dafermos)**.** *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ denote the maximal future development of smooth spherically symmetric asymptotically flat with two ends initial data for the Einstein-Maxwell-Klein-Gordon system with $\mathfrak{m}^2 = \mathfrak{e} = 0$ such that*

$$0 < |Q_+| < \varpi_+. \tag{8}$$

Then, $\mathcal{CH}_{i^+} \neq \emptyset$. Moreover, $(\mathcal{M}, g_{\mu\nu})$ is future-extendible as a C^0-Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}})$, which can be taken to be spherically symmetric, and there exists continuous functions $\widetilde{\phi}$ and $\widetilde{F_{\mu\nu}}$ defined on $\widetilde{\mathcal{M}}$ such that $\widetilde{\phi}$ and $\widetilde{F_{\mu\nu}}$ restricted to $\mathcal{M}$ coincide with ϕ and $F_{\mu\nu}$. In fact, condition (8) is satisfied for solutions arising from arbitrary initial data in a suitable open neighborhood of Reissner-Nordström initial data. In particular, the C^0-spherically symmetric formulation of strong cosmic censorship is false.

A solution of Theorem 1.4 has a Penrose diagram that admits an extension as depicted below.



To prove Theorem 1.4, Dafermos relies heavily on the decay properties of the scalar field along $\mathcal{H}^+$. This decay will be discussed in §1.4.3.

In order to highlight the importance of trapped surface formation to this discussion, we note that Dafermos also deduces the existence of a non-empty 'asymptotically connected' component of the outermost apparent horizon[23] $\mathcal{A}'$ that terminates at i^+ (cf. Williams [67]). This is given in

[21] In the case $\mathfrak{e} \neq 0$, see §2.2.2.

[22] Because $\mathfrak{e} = 0$, this can be taken to mean, without loss of generality, 'as measured along $\mathcal{H}^+$', since Q_+ is *globally* constant (cf. footnote 26).

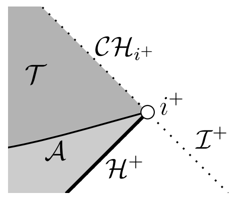
[23] The outermost apparent horizon $\mathcal{A}' \subset \mathcal{A}$ is a (possibly empty, not necessarily connected) achronal curve defined by the set of all $p \in \mathcal{A}$ whose past-directed ingoing null segment in $\mathcal{Q}^+$ lies in the regular region $\mathcal{R}$ with at least one $q \in \mathcal{R} \cap J^-(\mathcal{I}^+)$.

Theorem 1.5 (Dafermos)**.** *Let* $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ *denote the maximal future development of initial data as in Theorem 1.4. Then, there exists a non-empty 'asymptotically connected' component of the outermost apparent horizon* $\mathcal{A}'$ *that terminates at* i^+*. Moreover, in a sufficiently small neighborhood* $\mathcal{U} \subset \mathbb{R}^{1+1}$ *of* i^+*,*

$$\mathcal{A}' \cap \mathcal{U} = \mathcal{A} \cap \mathcal{U},$$

and, in particular,

$$I^+(\mathcal{A} \cap \mathcal{U}) \cap \mathcal{Q}^+ \subset \mathcal{T}.$$



To prove Theorem 1.4, it is necessary to first establish Theorem 1.5. Although the role of trapped surface formation is very different, this should be reminiscent of Theorems 1.2 and 1.3: Deducing that $\mathcal{A}$ has a limit point on i^+ is necessary to prove the *stability* of the Cauchy horizon $\mathcal{CH}_{i^+}$, as opposed to deducing that $\mathcal{A}$ has a limit point on b_Γ to prove the *instability* of the central Cauchy horizon $\mathcal{CH}_\Gamma$.

Of course, the model of Dafermos does not admit central Cauchy horizons, nor does the model of Christodoulou admit non-trivial $\mathcal{CH}_{i^+}$, but the analogy is interesting. Within the context of the more general Einstein-Maxwell-Klein-Gordon system, this tantalizing behavior, linking both cosmic censorship conjectures to trapped surface formation, can be further explored since both types of Cauchy horizons can be admitted. Indeed, an analogue of Theorem 1.5 has already been shown by the author when the topology of the initial data has one asymptotically flat end. See Theorem 1.10 in §1.6.2.

1.4.3 'No-hair theorem' and Price's law

In the study of gravitational collapse, one may ask: What are the possible 'end-states' of evolution?

So-called 'no-hair theorems', e.g. as given, in the present context, by Mayo and Bekenstein in [53], assert that if a spherically symmetric Einstein-Maxwell-Klein-Gordon black hole spacetime $(\mathcal{M}, g_{\mu\nu})$ is, in addition, stationary, i.e. the spacetime admits a Killing vector field that is asymptotically timelike in a neighborhood of $\mathcal{I}^+$, then $(\mathcal{M}, g_{\mu\nu})$ is a member of the Reissner-Nordström family.

For dynamic spacetimes as given in Theorem 1.1, if $\mathcal{BH} \neq \emptyset$ and the exterior geometry 'settles down' so as to give rise to a black hole spacetime that is asymptotically stationary as i^+ is approached, then the above 'no-hair theorem' suggests that the spacetime approaches Reissner-Nordström. The quantitative study of this decay ('settle down') mechanism is associated with the name of Price.

Formulated in [59], Price postulates that (massless) gravitational radiation decays polynomially with respect to the (asymptotically stationary) time co-ordinate along timelike surfaces of constant r. Later, the work of Gundlach et al. [36] refined the heuristics so as to postulate that (massless) gravitational flux along the event horizon (resp. future null infinity) will have polynomial decay with respect to a suitable advanced (resp. retarded) time co-ordinate.

In and of itself a major open problem, this decay mechanism, which we call here Price's law, is rigorously established by Dafermos and Rodnianski in the case $\mathfrak{m}^2 = \mathfrak{e} = 0$, provided that the black hole is 'sub-extremal in the limit' [33]. This is summarized in

Theorem 1.6 (Dafermos and Rodnianski)**.** *Let* $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ *denote the maximal future development of compactly-supported spherically symmetric asymptotically flat initial data for the Einstein-Maxwell-Klein-Gordon system as in Theorem 1.1 or 1.4 with* $\mathfrak{m}^2 = \mathfrak{e} = 0$*. Assume that* $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$*. If*

$$0 \leq |Q_+| < \varpi_+, \tag{9}$$

then, for all $\epsilon > 0$*, the scalar field* ϕ *satisfies*

$$|\phi| + |\partial_v \phi| \leq C_\epsilon v^{-3+\epsilon} \tag{10}$$

along $\mathcal{H}^+$*, where* v *is a suitable normalized advanced time co-ordinate.*[24]

We remark that the 'sub-extremal in the limit' condition (9) is satisfied for all black hole solutions arising in the model of Christodoulou.[25]

A generalization of Price's law to the case $\mathfrak{m}^2 \neq 0$ and $\mathfrak{e} \neq 0$ will be discussed in §1.5.2.

1.4.4 'Mass inflation' and strong cosmic censorship

While Theorem 1.6 gives an upper bound for the decay of a massless real-valued scalar field, heuristic analysis [2, 3, 35, 59] and numerical studies [6, 36] suggest that, generically, there is a similar lower bound. In fact, the existence of such a generic lower bound may yet, in light of Theorem 1.4, prove significant in redeeming the validity of strong cosmic censorship, for Dafermos shows that if, indeed, such a lower bound for decay holds along $\mathcal{H}^+$ for any 'sub-extremal in the limit' black hole, then the curvature must blow up along $\mathcal{CH}_{i^+}$ [26]. This provides mathematical confirmation of the 'mass-inflation' scenario of Israel and Poisson [58]. This is given in

Theorem 1.7 (Dafermos)**.** *Let* $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ *be as in Theorem 1.4, where, in addition, for a suitably normalized advanced time co-ordinate* v *and a large* V*,*

$$|\partial_v \phi| \geq C v^{-9+\epsilon} \qquad \textit{for all} \quad v \geq V \tag{11}$$

along $\mathcal{H}^+$*, for some* $\epsilon > 0$*. Then, the curvature blows up along* $\mathcal{CH}_{i^+}$*.*

If (11) can be shown to hold for generic initial data, then the C^2-spherically symmetric formulation of strong cosmic censorship is *true*! We state this in

Corollary 1.1. *Let* $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ *be as in Theorem 1.7. If (11) holds for generic initial data, then, in particular, the* C^2*-spherically symmetric formulation of strong cosmic censorship is* true*.*

1.5 Conjectures

Generalizing the results of Christodoulou and Dafermos to the full Einstein-Maxwell-Klein-Gordon system is, needless to say, no easy task. We discuss a few conjectures here to put forthcoming results, announced in §1.6, into the proper context. For convenience, we will here formulate all conjectures in the context of smooth developments as in Theorem 1.1. This being said, experience with the model of Christodoulou (cf. §1.4.1) indicates that it may be more natural to consider a larger class of solutions. The reader may wish to replace the smooth initial data and their maximal development in the statement of the conjectures with less regular initial data and their maximal development for which the conclusion of Theorem 1.1 can still be shown. See also the discussion of regularity in §1.8.7.

[24] Decay is also established along $\mathcal{I}^+$ and timelike curves of constant r, but only the decay along $\mathcal{H}^+$ is directly relevant for cosmic censorship. We shall, therefore, only make reference to decay along $\mathcal{H}^+$ in what follows.

[25] One can deduce this, *a posteriori*, from the statement of Theorem 1.2. Since, in this case, $\mathcal{CH}_{i^+} = \emptyset$, if $\mathcal{BH} \neq \emptyset$, then, necessarily, $\mathcal{S} \neq \emptyset$, whence $\mathcal{A} \neq \emptyset$ and $\varpi_+ \geq \inf_{\mathcal{A}} \frac{1}{2} r > 0 = Q_+$. In establishing Theorem 1.2, however, (9), i.e. $\varpi_+ > 0$, must first be shown (cf. [14]).

1.5.1 'Sub-extremal in the limit' black holes

The Einstein-Maxwell-Klein-Gordon system admits 'extreme' black hole solutions. In [1], Aretakis shows that the wave equation exhibits both stability and instability properties on extreme Reissner-Nordström horizon geometries, suggesting that the analysis required to deal with the class of 'extreme' black hole spacetimes will be delicate. That having been said, the following conjecture would imply that 'extreme' black hole solutions are non-generic and thus, in particular, one can ignore them in the context of the study of cosmic censorship.

Conjecture 1.3 (**'Sub-extremality' Conjecture**). *For the development of generic initial data as in Theorem 1.1, if $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$, then the black hole is 'sub-extremal in the limit'.*[26]

It should be noted that the asymptotic charge Q_+ (and hence the re-normalized mass ϖ_+) of the black hole is not, *a priori*, well-defined when $\mathfrak{e} \neq 0$. Moreover, in this case, unlike the case $\mathfrak{e} = 0$ in which (the topological) charge Q_+ is globally constant, it may be possible that the charge radiates completely to infinity. We, however, make the following reasonable conjecture.

Conjecture 1.4 (**Non-vanishing Charge Conjecture**). *For the development of generic initial data as in Theorem 1.1 with $\mathfrak{e} \neq 0$, if $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$, then the asymptotic charge Q_+ of the black hole is non-zero.*

This conjecture is relevant in view of (8).

1.5.2 Price's law

With respect to the decay rate (10) in Theorem 1.6, numerical work [5, 38] suggests that the massive charged scalar 'hairs' of a black hole will decay slower than the massless neutral (real) ones. In particular, the following conjecture of (a version of) Price's law appears reasonable.

Conjecture 1.5 (**Price's Law Conjecture**). *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ be as in Theorem 1.1. Assume that $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$. If the black hole is 'sub-extremal in the limit', then the scalar field ϕ satisfies, where v is a suitably normalized advanced time co-ordinate,*

$$|\phi| + |\partial_v \phi| \leq C v^{-p}$$

along $\mathcal{H}^+$, for $p = \frac{5}{6}$ if $\mathfrak{m}^2 \neq 0$, or for p satisfying $1 \leq p < 2$ if $\mathfrak{m}^2 = 0$.[27]

1.5.3 Trapped surface formation

As discussed in §1.4.1, trapped surface formation is central to establishing that, generically, $\mathcal{CH}_\Gamma = \emptyset$. Given the nature of the argument sketched in §1.4.1, Christodoulou was led to a trapped surface conjecture in [22], which, in the context of spherical symmetry, takes the form of

Conjecture 1.6 (**Spherical Trapped Surface Conjecture**). *For generic initial data as in Theorem 1.1, the maximal future development either has $b_\Gamma = i^+$ or $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ has a limit point on b_Γ, whence,* a fortiori*, $\mathcal{CH}_\Gamma = \emptyset$.*

By Statement III of Theorem 1.1, Conjecture 1.1 follows from Conjecture 1.6. Moreover, in the case $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$, by Statement VII of Theorem 1.1, Conjecture 1.6 also implies

[26] We emphasize that 'sub-extremal in the limit' is to be understood in some neighborhood of i^+ in $J^-(i^+)$.

[27] When the scalar field is massless and charged, the expected fall-off rate p is dependent on the dimensionless parameter $|\mathfrak{e}Q_+|$ (cf. [38]).

(the C^0-formulation[28] of) Conjecture 1.2 since $\mathcal{CH}_{i^+} = \emptyset$. More generally (see Statement VII of Theorem 1.1), if Conjecture 1.6 were true, then the problem of strong cosmic censorship completely reduces to understanding the behavior of the solution near $\mathcal{CH}_{i^+}$. In short, implicit in Conjecture 1.6 is a partial result concerning strong cosmic censorship.

If, however, we consign ourselves to just resolving weak cosmic censorship, then Theorem 1.1 actually allows us to state a *weaker* trapped surface conjecture, from which weak cosmic censorship would also follow. Indeed, since the presence of a single (marginally) trapped surface indicates[29] that a spacetime has a non-empty black hole region, Theorem 1.1 immediately gives, *a fortiori*,

Corollary 1.2. *Under the assumptions of Theorem 1.1, if $\mathcal{A} \neq \emptyset$, then $\mathcal{I}^+$ is complete.*

Conjecture 1.1 then follows from

Conjecture 1.7 (**Weak Spherical Trapped Surface Conjecture**). *For generic initial data as in Theorem 1.1, the maximal future development has either $b_\Gamma = i^+$ or $\mathcal{A} \neq \emptyset$.*

In delimiting the geometry of the trapped region (cf. Theorem 1.5), we also state

Conjecture 1.8 (**Outermost Apparent Horizon Conjecture**). *For initial data as in Theorem 1.1, if the maximal future development has $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$ and the black hole is 'sub-extremal in the limit', then there exists a non-empty 'asymptotically connected' component of the outermost*[30] *apparent horizon $\mathcal{A}'$ that terminates at i^+. Moreover,*

$$\mathcal{A}' \cap \mathcal{U} = \mathcal{A} \cap \mathcal{U}$$

in a sufficiently small neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of i^+ and, in particular,

$$I^+(\mathcal{A} \cap \mathcal{U}) \cap \mathcal{Q}^+ \subset \mathcal{T}.$$

By Statement IV of Theorem 1.1, Conjecture 1.8, in particular, implies that r extends continuously to $\mathcal{CH}_{i^+}$ in a sufficiently small neighborhood of i^+.

If, in addition to Conjecture 1.8, Conjecture 1.6 holds, then $\mathcal{B}^+ \backslash \left(i^+ \cup \mathcal{I}^+ \cup i^0\right)$ is always 'preceded' by a trapped region. This scenario should be compared with the assumptions of the trapped surface conjecture given by Christodoulou in [22]. We see that, in the terminology of [22], under Conjecture 1.8, the terminal indecomposable past sets $I^-(p) \cap \mathcal{Q}^+$ for $p \in \mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+}$ would correspond to sets whose trace on Σ do *not* have compact closure, but which would nonetheless satisfy Christodoulou's condition for containing a trapped surface.

1.5.4 Cauchy horizon conjectures

In light of the results discussed in §1.4.2, it seems reasonable to conjecture the following.

Conjecture 1.9 (**Continuous Extendibility Conjecture**). *For the development of initial data as in Theorem 1.1, if $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$, the black hole is 'sub-extremal in the limit' and the asymptotic charge $Q_+ \neq 0$, then $\mathcal{CH}_{i^+} \neq 0$ and the metric is continuously extendible beyond $\mathcal{CH}_{i^+}$.*

If the set of initial data for which the assumptions of Conjecture 1.9 hold has non-empty interior, then the C^0-formulation of Conjecture 1.2 is *false*!

Not all hope is lost for the fate of strong cosmic censorship, though. For, if Conjecture 1.6 can be established, then the C^2-formulation of Conjecture 1.2 reduces to showing that

[28] cf. discussion of regularity in footnote 15.

[29] The converse is not true. A black hole region need not contain a trapped surface. Note, however, Conjecture 1.8.

[30] See footnote 23 for a definition.

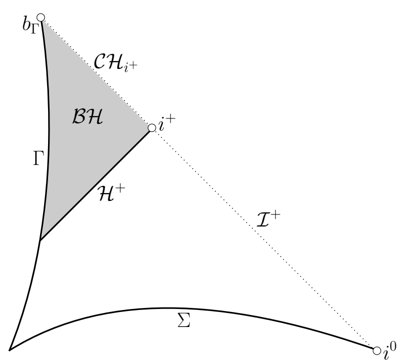
the curvature blows up along $\mathcal{CH}_{i^+}$. Because it is expected that a complex-valued scalar field will have late-time oscillatory behavior along $\mathcal{H}^+$ (suggested heuristically in [38]), it seems unlikely, as a result, that the lower bound (11) of Theorem 1.7 will hold. Heuristics, nonetheless, suggest that 'mass inflation' still occurs and it is reasonable to expect that the conclusion of Theorem 1.7 is true. We, therefore, state

Conjecture 1.10 ($\mathcal{CH}_{i^+}$ **Curvature Blow-up Conjecture**). *For generic initial data as in Theorem 1.1, if the maximal future development has $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$, then the curvature blows up along $\mathcal{CH}_{i^+}$.*

Although not relevant to weak and strong cosmic censorship, one might expect that null boundary components on which $r = 0$, if they occur at all, to be unstable. As such, the following conjecture seems reasonable.

Conjecture 1.11 (**Spacelike** $r = 0$ **Singularity Conjecture**). *For generic initial data as in Theorem 1.1, if $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$, then $\mathcal{S}^1_\Gamma \cup \mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+} = \emptyset$, $\mathcal{S} \neq \emptyset$ and $\mathcal{S}$ is C^1-spacelike.*

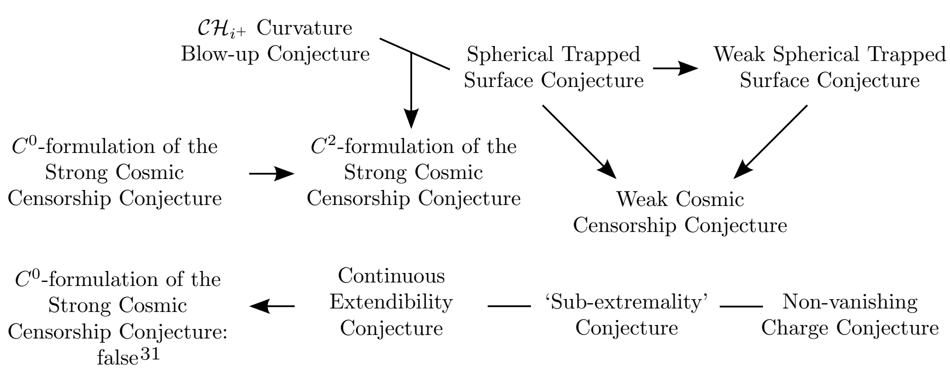
We should remark that although black hole solutions without a spacelike singularity, as depicted in diagram VII below, do not serve as counter-examples to weak and strong cosmic censorship, it is reasonable to conjecture, as above, that they would be ruled out, nonetheless, by genericity, hence we have included the statement $\mathcal{S} \neq \emptyset$ in the above Conjecture.



VII: black hole without a spacelike singularity

1.5.5 Web of implications: cosmic censorship

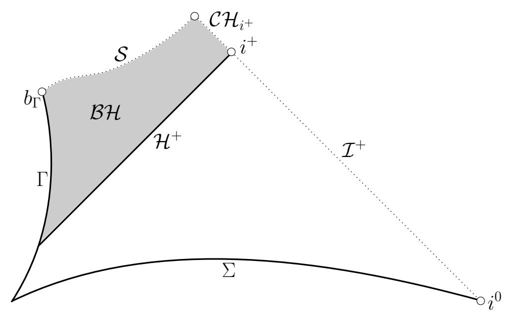
For convenience and clarity, we collect the various conjectures and their implications in regards to weak and strong cosmic censorship below.



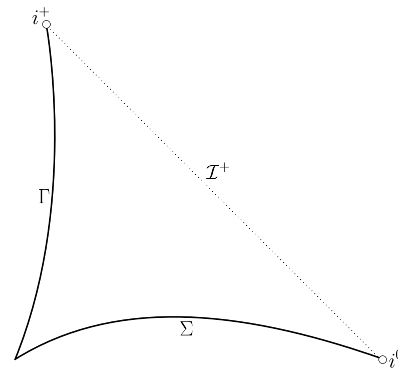
[31] In showing that the C^0-formulation of strong cosmic censorship is *false*, it suffices to show a weaker version of Conjectures 1.3 and 1.4, for the statements of the latter would only need to hold on an open set of solutions, not for generic solutions.

1.5.6 Generic Einstein-Maxwell-Klein-Gordon spacetimes

We wish to conclude this section with a summarized description of *generic* spherically symmetric Einstein-Maxwell-Klein-Gordon spacetimes, as would follow from a positive resolution to Conjectures 1.3–1.11. The resulting two classes of generic solutions, which we shall call the black hole case and the non-black hole case, have Penrose diagrams as depicted below.



VIII: black hole with a NFSE Cauchy horizon



IX: dispersive or 'star-like'

The black hole case

Conjectured generic Einstein-Maxwell-Klein-Gordon black hole solutions would have Penrose diagram depicted in diagram VIII and would have the following properties:

1. the black hole is 'sub-extremal in the limit';
2. the asymptotic charge Q_+ is well-defined and $Q_+ \neq 0$;
3. in a neighborhood of i^+ in $J^-(\mathcal{I}^+)$ the spacetime asymptotically approaches Reissner-Nordström at a rate given by Price's law;
4. $\mathcal{S} \neq \emptyset$ (hence $\mathcal{A} \neq \emptyset$) and $\mathcal{S}$ is C^1-spacelike;
5. $\mathcal{A}$ has limit points on b_Γ and i^+;
6. $\mathcal{S}^1_\Gamma \cup \mathcal{S}^2_\Gamma \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_{i^+} = \emptyset$; and,
7. $\mathcal{CH}_{i^+} \neq \emptyset$ and the curvature blows up on $\mathcal{CH}_{i^+}$.

The non-black hole case

Conjectured generic Einstein-Maxwell-Klein-Gordon non-black hole solutions would have Penrose diagram as depicted in diagram IX and are of two possible types, one of which we shall call dispersive and the other 'star-like'.

In the case $\mathfrak{m}^2 = 0$, it is expected, as in the model of Christodoulou, that all non-black hole spacetimes will have vanishing final Bondi mass M_f. These solutions will be called dispersive. By the results of Chae [7], there is an 'open' set of initial data containing trivial data whose developments are dispersive.

On the other hand, when $\mathfrak{m}^2 \neq 0$, the Einstein-Maxwell-Klein-Gordon system can, in general, admit charged (boson) star solutions [44, 63], i.e. when $\mathfrak{m}^2 \neq 0$, the scalar field may not be 'dispersive'; we shall call such solutions 'star-like'. Let us also note, however, that the final Bondi mass M_f is not a suitable measure of dispersive phenomena, as massive scalar fields do not radiate to $\mathcal{I}^+$ (cf. [37]).

The rich possibility of solutions in the $\mathfrak{m}^2 \neq 0$ case will complicate the analysis of non-black hole solutions, but its scope lies outside the context of this paper. Indeed, one may formulate a host of conjectures, as we did in the black hole case, regarding the properties of generic non-black hole solutions, but expounding on such properties would here take us too far afield.

1.6 Forthcoming results

In a series of forthcoming papers, we will expand on the characterization given in Theorem 1.1 and partially address the conjectures presented in §1.5, but we wish already in the present paper to highlight some important results.

1.6.1 Trapped surface formation

We have already discussed the significant role trapped surface formation may have in settling the cosmic censorship conjectures. It is not, *a priori*, clear, however, if trapped surfaces are able to form dynamically in evolution.[32] In a forthcoming paper [49], the author shows that there is, indeed, a 'large' class of regular initial data, i.e. containing neither marginally trapped nor trapped surfaces, whose future development contains a marginally trapped surface. This is summarized in

Theorem 1.8 (Kommemi [49])**.** *For a scalar field of arbitrary mass $\mathfrak{m}^2 \geq 0$ and arbitrary charge $\mathfrak{e}$, there are solutions to the spherically symmetric Einstein-Maxwell-Klein-Gordon system with Penrose diagram as in Theorem 1.1 such that $\mathcal{A} \neq \emptyset$ but $\Sigma \cap \mathcal{A} = \emptyset$ (in fact, $\Sigma \cap \mathcal{BH} = \emptyset$).*

In proving Theorem 1.8, as in the monumental work of Christodoulou on trapped surface formation in vacuum [24], which has also recently been extended by Klainerman and Rodnianski [46, 47], we consider initial data that are trivial along C_p^-. If data, which may be *arbitrarily dispersed*, are nonetheless suitably 'large', the emergence of a marginally trapped surface is guaranteed to occur in their future development. A sufficient largeness condition is characterized for a given annular region of spacetime whose dimensionless size and mass content satisfy[33]

$$1 > \text{mass content} \gtrsim \sqrt{\text{size}}.$$

This should be compared with the result given in Theorem 1.3 for $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$.

1.6.2 Cauchy horizon stability and C^0-extendibility

In another forthcoming paper [48], the author further characterizes the boundary of spherically symmetric Einstein-Maxwell-Klein-Gordon spacetimes by showing that if the scalar field decays suitably along an event horizon of a black hole that is 'sub-extremal in the limit', then the Cauchy horizon $\mathcal{CH}_{i^+}$ of Theorem 1.1 is non-empty and the maximal future development is continuously extendible beyond $\mathcal{CH}_{i^+}$ (cf. Conjecture 1.9). In particular, we have

[32] This should be compared with Proposition 3.1, which states that anti-trapped surfaces are, in fact, *non*-evolutionary (cf. [18]).

[33] A mass content greater than one would indicate the presence of a trapped surface, and mass content equal to one in the case of a marginally trapped surface.

Theorem 1.9 (Kommemi [48]). *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ denote the maximal future development of spherically symmetric asymptotically flat initial data as in Theorem 1.1. Assume that $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$. If the black hole is 'sub-extremal in the limit', the asymptotic charge $Q_+ \neq 0$ and the scalar field ϕ satisfies, for p sufficiently large,*[34]

$$|\phi| + |\partial_v \phi| \leq C v^{-p} \tag{12}$$

along $\mathcal{H}^+$, where v is a suitably normalized advanced time co-ordinate, then

$$\mathcal{CH}_{i^+} \neq \emptyset.$$

Moreover, $(\mathcal{M}, g_{\mu\nu})$ is future-extendible as a C^0-Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}})$, which can be taken to be spherically symmetric, and there are continuous functions $\widetilde{\phi}$ and $\widetilde{F_{\mu\nu}}$ defined on $\widetilde{\mathcal{M}}$ such that $\widetilde{\phi}$ and $\widetilde{F_{\mu\nu}}$ restricted to $\mathcal{M}$ coincide with ϕ and $F_{\mu\nu}$.

The proof of Theorem 1.9 relies on a detailed understanding of the interior black hole geometry, which includes regions of infinite red-shift near the event horizon and regions of infinite blue-shift near the Cauchy horizon. While similar in spirit to the proof of Theorem 1.4, the analysis required to prove Theorem 1.9 must overcome two complications that preclude the presence of special monotonicity heavily exploited in [26]: (1) $\mathfrak{m}^2$ is not assumed to vanish; and, (2) the Maxwell field $F_{\mu\nu}$ does not decouple from the rest of the system when $\mathfrak{e} \neq 0$.

In the course of establishing the result, we also deduce the existence of a non-empty 'asymptotically connected' outermost apparent horizon $\mathcal{A}'$ terminating at i^+ (cf. Theorem 1.5 and Conjecture 1.8). This is summarized in

Theorem 1.10 (Kommemi [48]). *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ denote the maximal future development of spherically symmetric asymptotically flat initial data as in Theorem 1.1. Assume that $\mathcal{Q}^+ \backslash J^-(\mathcal{I}^+) \neq \emptyset$. If the black hole is 'sub-extremal in the limit' and the scalar field ϕ satisfies, for all $\epsilon > 0$,*

$$|\phi| + |\partial_v \phi| \leq C v^{-\frac{1}{2}-\epsilon}$$

along $\mathcal{H}^+$, where v is a suitably normalized advanced time co-ordinate, then there exists a non-empty 'asymptotically connected' component of the outermost apparent horizon $\mathcal{A}'$ that terminates at i^+. Moreover,

$$\mathcal{A}' \cap \mathcal{U} = \mathcal{A} \cap \mathcal{U}$$

in a sufficiently small neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of i^+ and, in particular,

$$I^+(\mathcal{A} \cap \mathcal{U}) \cap \mathcal{Q}^+ \subset \mathcal{T}.$$

Provided the black hole is 'sub-extremal in the limit', we have therefore shown that Conjecture 1.8 follows from a weaker version of Price's law (Conjecture 1.5).

1.7 Generalized extension principle

The main content of Theorem 1.1 consists of establishing an extension principle, characterizing 'first singularities', considerably stronger than that proposed by Dafermos in [28]. While useful for weak cosmic censorship, the extension principle of [28], which concerns only the closure of the regular region $\mathcal{R}$ of spacetime, is insufficient to delve into the inner reaches of the black hole region where there are, potentially, trapped surfaces. Not only will we prove, in particular, the extension principle of [28] for our system (2)–(6), but we will give a stronger

[34] We currently require $p > 2$ in (12). In light of the conjectured Price's law (cf. §1.5.2) we would hope to improve to some $p < 2$. One can, nevertheless, for arbitrary p, construct non-trivial asymptotically flat solutions satisfying the assumptions of Theorem 1.9 by solving a scattering problem.

result: A 'first singularity' must emanate from a spacetime boundary to which the area-radius function r extends continuously to zero.

An important result in its own right, we wish to include this generalized extension principle as a stand-alone statement. In view of applications to cosmological topologies or to the case of two asymptotically flat ends, it is useful to add an assumption on the finiteness of the spacetime volume, which, as we shall see (cf. Proposition 3.2), can be retrieved under the assumptions of Theorem 1.1. We thus formulate the extension principle as follows:

Theorem 1.11. *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ denote the maximal future development of smooth spherically symmetric initial data for the Einstein-Maxwell-Klein-Gordon system. For $p \in \overline{\mathcal{Q}^+}$ and $q \in (I^-(p) \cap \mathcal{Q}^+) \setminus \{p\}$ such that $\mathcal{D} = (J^+(q) \cap J^-(p)) \setminus \{p\} \subset \mathcal{Q}^+$, if*

1. *$\mathcal{D}$ has finite spacetime volume; and,*
2. *there are constants r_0 and R such that*

$$0 < r_0 \leq r(p') \leq R < \infty \qquad \textit{for all } p' \in \mathcal{D},$$

then $p \in \mathcal{Q}^+$.

We should re-iterate that there is neither an assumption on the global (say, asymptotically flat or hyperboloidal) geometry nor the topology of the initial data in Theorem 1.11. This generality of the extension principle is made possible by the fact that the proof of Theorem 1.11 does *not* rely on energy flux conservation[35], which is unavailable in the trapped region, but that it directly exploits the special null structure in the Einstein-Maxwell-Klein-Gordon system. This null structure manifests itself as follows: To pointwise control Ω^2, ϕ and $F_{\mu\nu}$, it suffices to give spacetime integral estimates

$$\int\int r^2 T_{uv} \, \mathrm{d}u\mathrm{d}v \qquad \text{and} \qquad \int\int \mathrm{D}_u\phi\,(\partial_v\phi)^\dagger + \partial_v\phi\,(\mathrm{D}_u\phi)^\dagger \;\, \mathrm{d}u\mathrm{d}v. \tag{13}$$

In particular, that the 'bad' uu- and vv-components do not appear in the integrands of (13) is a consequence of the null structure (both of the coupling of the matter equations to gravity and the matter equations themselves, respectively). This allows us to integrate by parts (13) so as to always exploit one of the 'good' ingoing or outgoing directions. The symmetrization in

$$\mathrm{D}_u\phi\,(\partial_v\phi)^\dagger + (\mathrm{D}_u\phi)^\dagger\,\partial_v\phi$$

plays an important role in being able to make use of the null structure. See §4.3.1–4.3.6.

1.8 General spherically symmetric Einstein-matter systems

Because of the importance of a suitable extension principle in providing a global characterization of spacetime, we wish to cast the contents of [28] and Theorem 1.11 in a much greater context.

1.8.1 Weak and generalized extension principles

We begin with the following definitions, recalling the notation introduced in Theorem 1.1.

Weak extension principle. The *weak extension principle* is satisfied for an Einstein-matter system if the following condition holds: Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \ldots)$ denote the maximal future development of spherically symmetric asymptotically flat initial data with one end containing no anti-trapped regions. Suppose $p \in \overline{\mathcal{R}} \backslash \overline{\Gamma} \subset \overline{\mathcal{Q}^+}$ and $q \in \left(I^-(p) \cap \overline{\mathcal{R}}\right) \setminus \{p\}$ are such that $(J^-(p) \cap J^+(q)) \setminus \{p\} \subset \mathcal{R} \cup \mathcal{A}$. Then, $p \in \mathcal{R} \cup \mathcal{A}$.

[35] provided by the monotonicity of the Hawking mass

We emphasize that the closure and causal-geometric constructions are with respect to the topology of the ambient $\mathbb{R}^{1+1}$. The weak extension principle states that a 'first singularity' emanating from the regular region can only do so from the center.

Generalized extension principle. The *generalized extension principle* is satisfied for an Einstein-matter system if the following condition holds: Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \ldots)$ denote the maximal future development of spherically symmetric initial data. For $p \in \overline{\mathcal{Q}^+}$ and $q \in (I^-(p) \cap \mathcal{Q}^+) \setminus \{p\}$ such that $\mathcal{D} = (J^+(q) \cap J^-(p)) \setminus \{p\} \subset \mathcal{Q}^+$, suppose that

1. $\mathcal{D}$ has finite spacetime volume; and,
2. there are constants r_0 and R such that

$$0 < r_0 \leq r(p') \leq R < \infty \qquad \text{for all } p' \in \mathcal{D}.$$

Then, $p \in \mathcal{Q}^+$.

The generalized extension principle states that given a 'first singularity' either it must emanate from a spacetime boundary to which the area-radius function r can be extended continuously to zero, or else the causal past of the 'first singularity' will have infinite spacetime volume.

As our naming convention suggests, the generalized extension principle should imply the weak extension principle. Without, however, a suitable energy condition this need not be the case. We show that the null energy condition is sufficient to rule out the possibility of the causal past of the 'first singularity' having infinite spacetime volume whenever the initial data are free of anti-trapped regions (cf. Proposition 3.2 in §3.2).

As in Theorem 1.11, the generalized extension principle is stated without reference to the topology or geometry of the initial data and can be applied, for example, to the cosmological setting or the case with two asymptotically flat ends.

1.8.2 'Tame' matter models

In accordance with the above extension principles, we introduce the following notions of 'tame' Einstein-matter systems.

Definition 1. A spherically symmetric Einstein-matter system obeying the dominant energy condition that satisfies the weak extension principle is called *weakly tame*.

Definition 2. A spherically symmetric Einstein-matter system obeying the dominant energy condition that satisfies the generalized extension principle is called *strongly tame*.

With this classification we have

Proposition 1.1. *A strongly tame Einstein-matter system is weakly tame.*

For a proof of this statement, see §5.1.

1.8.3 Generalization of Theorem 1.1 to strongly tame matter models

The proof of Theorem 1.1, after the conclusion of Theorem 1.11 has been established, follows from a series of monotonicity arguments governed by the dominant energy condition.[36] No structure particular to the Einstein-Maxwell-Klein-Gordon system is used. As a result, we can state

[36] Much of Theorem 1.1, in fact, uses the monotonicity governed by Raychaudhuri's equation, which just needs the null energy condition (cf. the proof of Theorem 1.1 in §5).

Theorem 1.12. *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \ldots)$ denote the maximal future development of smooth spherically symmetric asymptotically flat with one end initial data for a strongly tame Einstein-matter system containing no anti-trapped spheres of symmetry. Then, the conclusion of Theorem 1.1 holds for this system.*

See the comment in §6 regarding the proof of this statement.

1.8.4 A version of Theorem 1.1 for weakly tame matter models

One can deduce from the proof of Theorem 1.1 that the weak extension principle, in fact, recovers the boundary characterization of Statement II except for the characterization that r vanishes on $\mathcal{S}$. In other words, establishing the weak extension principle is not sufficient to rule out the possibility that r has non-zero limit values on (part of) $\mathcal{S}$.

To establish many of the statements of Theorem 1.1, however, it is not important to have a characterization of r on $\mathcal{S}$; these results, consequently, hold, *mutatis mutandis*, for weakly tame matter models. In particular, we state

Theorem 1.13. *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \ldots)$ denote the maximal future development of smooth spherically symmetric asymptotically flat with one end initial data for a weakly tame Einstein-matter system containing no anti-trapped spheres of symmetry. Then, except for the statements enclosed in boxes, the conclusion of Theorem 1.1 holds for this system.*

It should be noted, moreover, that many of the enclosed 'boxed' statements can be (trivially) re-worked as to apply even in the weakly tame case:[37]

Statement IV.3* If $\mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+} \neq \emptyset$, then $\mathcal{A} \cup \mathcal{T} \neq \emptyset$. (If $\mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+} = \emptyset$, then $\mathcal{A} \cup \mathcal{T}$ is possibly empty.)

Statement IV.5a* If $\mathcal{A} \neq \emptyset$, then all limit points of $\mathcal{A}$ that lie on the boundary $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ lie on $\mathcal{CH}_{i^+} \cup i^+$ and on a (possibly degenerate) closed, necessarily connected interval of $b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma \cup \mathcal{S}$.

Statement IV.5e* If $\mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+} \neq \emptyset$, then $\mathcal{A}$ has a limit point on $b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma \cup \mathcal{S}$.

Statement VII.1* The Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is a continuous $[0, \infty]$-valued function on $\mathcal{Q}^+ \cup \mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+}$ that yields ∞ on $\mathcal{S}^2_\Gamma \cup \mathcal{S}_{i^+}$.

Statement VII.4* If $(\mathcal{M}, g_{\mu\nu})$ is future-extendible as a C^2-Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}})$, then there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting the manifold $\mathcal{M}$ such that the closure of the projection of $\gamma|_{\mathcal{M}}$ to $\mathcal{Q}^+$ intersects $\mathcal{CH}_\Gamma \cup \mathcal{S} \cup \mathcal{CH}_{i^+}$.

Since in a weakly tame model we know nothing, *a priori*, about the behavior of the metric at $\mathcal{S}$, we note that, in turn, Statement VII.4*, in practice, tells us very little about inextendibility properties. For this reason, establishing that an Einstein-matter system is strongly tame is a crucial first step in understanding strong cosmic censorship.

1.8.5 Examples of weakly and strongly tame models

We now give examples of known tame Einstein-matter systems.

[37] In the case of Statement VII.3, it is presumed that we can extend the solution into a neighborhood of i^+. Since, *a priori*, this neighborhood will contain trapped spheres, we must appeal to the generalized extension principle. Moreover, because we need to establish a positive (non-zero) lower bound on r in this neighborhood, although there is no explicit reference to $\mathcal{S}$, Statement VII.3 requires, indeed, that a characterization of r be given along $\mathcal{S}$.

Strongly tame

In the language of §1.8.2, Theorem 1.11 shows that Einstein-Maxwell-Klein-Gordon is strongly tame, whereas Dafermos and Rendall show in [32] that Einstein-Vlasov is also strongly tame.[38] We note that in both proofs one heavily exploits relevant null structure (arising, in the former case, from the coupling of the matter equations to gravity and the matter equations themselves, while in the latter case, just in the coupling). We make the following imprecise conjecture.

Conjecture 1.12. *If a spherically symmetric Einstein-matter system satisfies a suitable 'null condition', then the system is strongly tame.*

For a discussion of the 'null condition', see Klainerman [45].

Weakly tame

Dafermos shows in [27] that Einstein-Higgs with non-negative potential $V(\phi)$ is weakly tame.

Narita [54] considers 'first singularity' formation in the Einstein-wave map system with target $\mathbb{S}^3$ and H^2. In the language of the present paper, these models are weakly tame.

1.8.6 Exotic models

The definitions of weakly and strongly tame are tailored specifically so as to apply to classical, self-gravitating matter models. One often encounters in the physics literature, however, models that are 'exotic' in some respect. In the sequel, we will show that suitable notions of weakly and strongly tame can still be introduced for such exotic systems.

Exotic matter

Immediate from the proof of Theorem 1.11, it follows that Einstein-Klein-Gordon ($\mathfrak{e}^2 = F_{\mu\nu} = 0$) with $\mathfrak{m}^2 < 0$ satisfies the generalized extension principle, but because this non-classical matter model does not obey the dominant energy condition, it is not, according to our definition, strongly tame. The matter model, however, does obey the null energy condition. As the proof of Theorem 1.12 will make clear, many statements that hold for strongly tame Einstein-matter systems, in fact, follow from monotonicity governed by Raychaudhuri's equation, which just needs the null energy condition. As a result, much can still be said of the global structure of Einstein-Klein-Gordon spacetimes when $\mathfrak{m}^2 < 0$. See the recent work of Holzegel and Smulevici who consider spherically symmetric asymptotically AdS Einstein-Klein-Gordon spacetimes [40, 41].

In the case of Einstein-Higgs, the proof of the weak extension principle in [27] can be established with the help of the flux provided by the Hawking mass. The weak extension principle, consequently, can be given more generally for a potential that is bounded from below by a (possibly negative) constant: $V(\phi) \geq -C$. Allowing for such a lower negative bound, [27] was able to disprove a scenario of 'naked singularity' formation that had appeared in the high energy physics literature. Unless the potential is non-negative, however, this non-classical matter model is not weakly tame, by our definition, because it does not obey the dominant energy condition.

Although we will not discuss further such non-classical exotic matter, one could consider systems such as Einstein-Klein-Gordon (arbitrary $\mathfrak{m}^2$) as being 'quasi-strongly tame' and Einstein-Higgs with $V(\phi) \geq -C$ as being 'quasi-weakly tame', *et cetera*.

[38] Previously, Dafermos and Rendall had shown that Einstein-Vlasov is weakly tame in [30]. In view of Proposition 1.1, however, this is immediate from the subsequent work of [32].

Higher dimensional case

Dafermos and Holzegel show that the analogue of the weak extension principle holds for the maximal future development of asymptotically flat Einstein-vacuum initial data having triaxial Bianchi IX symmetry [29]. Here, spacetime $(\mathcal{M}, g_{\mu\nu})$ is 5-dimensional and $SU(2)$ acts by isometry. The Einstein-vacuum equations under this symmetry assumption can be written as a system of non-linear PDEs on a 2-dimensional Lorentzian manifold $\mathcal{Q}^+ = \mathcal{M}/SU(2)$ (possibly with boundary), whose 'two dynamical degrees of freedom' correspond to the possible deformations of the group orbit 3-spheres. This vacuum model shares a formal similarity with a spherically symmetric 4-dimensional Einstein-matter system, whose matter obeys the dominant energy condition. We can, as a result, formally view triaxial Bianchi IX as being weakly tame in our present context.

One can consider more straightforward generalizations to higher dimensions by considering spherically symmetric n-dimensional ($n \geq 4$) spacetimes $(\mathcal{M}, g_{\mu\nu})$ whereby $SO(n-1)$ acts by isometry (cf. [50]).

Modified gravity

Narita [55] has considered 'first singularity' formation in the spherically symmetric Einstein-Gauß-Bonnet-Klein-Gordon system ($n \geq 5$) with $\mathfrak{m}^2 = 0$. In the language of the present paper, this system is weakly tame.

1.8.7 Regularity of the maximal future development

Our notion of 'tameness' attempts to classify the type of 'first singularities' a given spherically symmetric Einstein-matter system will exhibit. This classification is naturally regularity-dependent. The discussion of weakly and strongly tame models above has been restricted to the case of smooth maximal future developments. It will often be convenient, even necessary, to consider developments that are non-smooth.

Solutions of bounded variation for the scalar field model

As we have mentioned before (cf. §1.4.1), in order to initiate the study of a large class of solutions sufficiently flexible to exploit the genericity assumption inherent in the formulation of cosmic censorship, Christodoulou introduces a notion of bounded variation (BV) solutions for the spherically symmetric Einstein-Klein-Gordon system with $\mathfrak{m}^2 = 0$. In [16], Christodoulou establishes the well-posedness of an initial value formulation of this system for given BV initial data. Christodoulou is, moreover, able to establish that the system is, in the language of the present paper, strongly tame.

Shell-crossing singularity formation in Einstein-dust

Consider the Einstein-Euler system with equation of state $p = 0$, i.e. a pressure-free fluid. This system is also known as Einstein-dust. The spherically symmetric (infinite dimensional family of asymptotically flat) solutions of the Einstein-dust system were first given by Tolman in [65] based on the work of Lemaître [51]. Beginning with the work of Oppenheimer and Snyder [56], which discussed in detail the gravitational collapse of a uniform density 'ball of dust', this matter model had (and still does) spawn great interest in the physics community.

In Yodzis et al. [68], it is shown that, in general, the Einstein-dust system forms ('naked') 'first singularities' away from the center, commonly referred to as shell-crossings, in the class of smooth maximal future developments. In the language of the present paper, this shows that Einstein-dust is *not* weakly tame (and hence not strongly tame) in the smooth sense. In a way, this result is unsurprising; shell-crossings occur already in the absence of gravity, e.g. on a fixed Minkowski background. It turns out, however, that the solution obtained

by extending beyond the shell-crossings makes physical sense (the metric is, in particular, still continuous [57] as long as $r > 0$). One can thus view Einstein-dust as being strongly tame in a suitable class of rough solutions. We note, however, the negative resolution of cosmic censorship for the Einstein-dust system, even in this more appropriate class, shown by Christodoulou (cf. §1.8.8).

Shock formation in Einstein-Euler

The breakdown of smooth solutions of the Euler system has been studied extensively in [8, 23, 64]. In coupling to gravity, Rendall and Ståhl [60] show that under assumption of plane symmetry, smooth solutions still break down in (arbitrarily short) finite time. It is believed that, as in the classical Euler system, the breakdown of Einstein-Euler again is a result of the discontinuities of the fluid flow, commonly referred to as shocks. In the language of the present paper, Einstein-Euler (for general equation of state) is *not* weakly tame (and hence not strongly tame) for smooth maximal future developments. Understanding the global properties of solutions to the Euler system, in the context of less regular developments, remains a long-standing (and very difficult!) open problem, where a large-data theory is unavailable even in $1+1$-dimensions.

The two-phase fluid model of Christodoulou

Christodoulou sought to understand a two-phase fluid model that would capture many of the features of actual stellar gravitational collapse and at the same time would be mathematically tractable. In [18], Christodoulou considers the spherically symmetric Einstein-Euler system with a two-phase barotropic equation of state given by

$$p = \begin{cases} 0 & \text{if } \rho \leq 1; \\ \rho - 1 & \text{if } \rho > 1, \end{cases}$$

i.e. the soft phase of the two-phase model coincides with that of dust ($p = 0$) while the hard phase coincides with that of a massless real-valued scalar field, where

$$p = \frac{1}{2}\left(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) - 1$$

with the restriction that $-g^{\mu\nu}\partial_\nu\phi$ be a future-directed timelike vector field.[39] This model seeks to capitalize on the insight gained via his study of dust (see below) and his subsequent work on scalar fields. Here, shocks develop from the development of smooth initial data in the form of the boundary between the phases. Their dynamics can then be understood as a free-boundary problem,[40] which is studied extensively in [19, 20, ...].[41] In this collection of work, Christodoulou shows, in particular, in the language of the present paper, that, in the context of a suitably rough maximal future development, this two-phase model is strongly tame.

1.8.8 Cosmic censorship for Einstein-dust and the two-phase model

In §1.8.7, we saw that the shell-crossing singularities of [68] are not 'naked singularity' solutions from the point of view of the correct concept of a solution. On the other hand, that, indeed, true 'naked singularities' generically form for the Einstein-dust model is later proven by Christodoulou [13]. Specifically, in the the collapse of an inhomogeneous ball of dust, there exists an open set of initial mass densities ρ_0 such that the mass density ρ will become infinite

[39] This requirement, i.e. ϕ is a time function, is necessary to ensure that the scalar field has a hydrodynamic interpretation.

[40] for which the timelike components of the phase boundary correspond to shocks

[41] Part of this series by Christodoulou is unpublished.

at some central point before the formation of a trapped surface occurs, as opposed to the shell-crossing 'first singularities' of [68] that are inessential. In the language of the present paper, the Penrose diagram of such a spacetime is given by diagram III in §1.4.1 ('light cone singularity') with $i^{\square} = i^{\text{naked}}$, where $\rho(b_\Gamma) = \infty$ and for which, generically, the metric is extendible across $\mathcal{CH}_\Gamma$. In particular, Christodoulou shows that Conjectures 1.1 and 1.2 are *false* for any suitable notion of an Einstein-dust solution.

It should be noted, however, that the work of Christodoulou on Einstein-dust is not the death knell of cosmic censorship. The equation of state $p = 0$ is a very special one, one which becomes less and less plausible as $\rho \to \infty$. If one wishes to consider the problem of cosmic censorship for Einstein-Euler, then more realistic equations of state must be allowed. In this sense, the two-phase model introduced by Christodoulou is perhaps the most tractable realistic model improving the pure dust case. Because of the scalar field structure of the hard phase, and in light of Theorem 1.2, it is reasonable to expect that cosmic censorship will be true; this conjecture, however, remains open.

1.8.9 Table of weakly tame and strongly tame models

We conclude this section with a summary of our discussion.

Unless otherwise noted by a box, the following hold for smooth maximal future developments. In the case of rough developments, the usual caveats about uniqueness apply, as the well-posedness statement, which is formulated 'downstairs', can only be discussed assuming the symmetry of the development.

For exotic models, a * indicates that weakly and strongly tame are to be understood in a suitable sense, e.g. in view of the failure of the energy condition (cf. §1.8.6).

Matter model	Weakly tame	Strongly tame	Remarks
Maxwell-Klein-Gordon	Yes	Yes	Theorem 1.11
Vlasov	Yes	Yes	[30, 32]
Klein-Gordon, $\mathfrak{m}^2 = 0$, $\boxed{\text{BV}}$	Yes	Yes	[16]
Higgs, $V(\phi) \geq 0$	Yes	?	[27]
Wave maps, target $\mathbb{S}^3$, H^2	Yes	?	[54]
Yang-Mills, et al.	?	?	null structure?
Euler ($p = 0$)	No	No	[68], shell-crossings
Euler ($p = 0$), $\boxed{\text{suitably rough}}$	Yes	Yes	[57], W.C.C. still *false*
Euler	No	No	[23, 60, 64], shocks
Euler, $\boxed{\text{suitably rough}}$	?	?	difficult open problem!
Two-phase fluid, $\boxed{\text{free-boundary}}$	Yes	Yes	[18, 19, 20, ...]
Vacuum, triaxial Bianchi IX	Yes*	?	[29], $n = 5$, SU(2)-action
Klein-Gordon, $\mathfrak{m}^2 < 0$	Yes*	Yes*	Theorem 1.11
Higgs, $V(\phi) \geq -C$	Yes*	?	[27]
Gauß-Bonnet-Klein-Gordon, $\mathfrak{m}^2 = 0$	Yes*	?	[55]

1.9 'Sharpness' of the boundary characterization

We briefly discuss here the 'sharpness' of the boundary characterization as given in Theorem 1.1.

For the purpose of this discussion, it is helpful to define a coarser description[42] of the boundary. Let us define

$$\widetilde{\mathcal{S}} = \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}.$$

[42]The set $\mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}$ can be understood as a unit in the statement of Theorem 1.1. See Statements IV.3, IV.5.e and VII.1. Indeed, $\mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}$ corresponds to the set $\mathcal{B}_s$ in [32]. The reason we choose to separate the components is to highlight the fact that r can be zero on the null components that emanate from b_Γ and $i^{\square}$.

The spacetime boundary (7) is then given by

$$\mathcal{B}^+ = b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma \cup \widetilde{\mathcal{S}} \cup \mathcal{CH}_{i^+} \cup i^\Box \cup \mathcal{I}^+ \cup i^0. \tag{14}$$

The sets b_Γ, $i^\Box$, $\mathcal{I}^+$ and i^0 are always non-empty. The set $\widetilde{\mathcal{S}}$ is non-empty for every black hole solution in the case $\mathfrak{m}^2 = \mathfrak{e} = F_{\mu\nu} = 0$ (cf. §1.4.1). Christodoulou [16, 17] constructs explicit examples in the BV class for which $\mathcal{S}^1_\Gamma$ is non-empty and examples for which $\mathcal{CH}_\Gamma$ is non-empty. On the other hand, an easy pasting argument involving the Reissner-Nordström solution yields examples in which $\mathcal{CH}_{i^+}$ is non-empty. Thus, we have the following theorem.

Theorem 1.14. *For each type of boundary component in (14), there exists a development of data as in Theorem 1.1 (possibly in the more general BV class) for which that component-type is non-empty.*

It may be useful to introduce the following nomenclature:

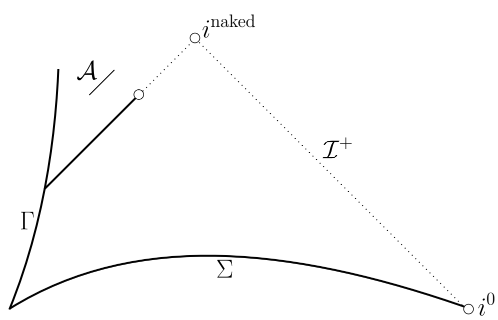
Definition 3. A strongly tame Einstein-matter model for which the conclusion of Theorem 1.14 holds is called *fully general*.

In this language, the Einstein-Maxwell-Klein-Gordon system is fully general, in fact, the 'simplest' model that is known to be fully general.[43] This is one way to understand the importance of the charged scalar field model in studying spherically symmetric formulations of cosmic censorship.

1.10 Is it possible to 'super-charge' a black hole?

Lastly, we would like to conclude with an immediate consequence of Theorem 1.1, which is of relevance to a discussion in the physics community.

The notion of 'destroying' the event horizon by means of 'super-charging' or 'super-spinning' (near-)extremal Reissner-Nordström or Kerr-Newman black holes has been entertained in the literature, e.g. [4, 9, 34, 42, 43, 52, 61, 62, 66], 'transforming' a black hole into a 'naked singularity'. If this were possible, then weak cosmic censorship would be false. These constructions, however, share the feature that $\mathcal{A} \neq \emptyset$. One is thus to imagine a Penrose diagram as depicted below:



In view of Theorem 1.1, such spactimes simply do not exist. Since $\mathcal{A} \neq \emptyset$, by Statement III of Theorem 1.1, $\mathcal{I}^+$ is complete and the correct Penrose diagram is as depicted in Theorem 1.1 with $\mathcal{BH} \neq \emptyset$ and $i^\Box = i^+$. Interestingly, one need not understand the precise long-time behavior of $\mathcal{H}^+$ to infer this. In particular, no 'naked singularity' can be created in the present model by 'super-charging' a black hole. This confirms the original intuition of Wald [66].

[43] One could, of course, define alternative notions of 'fully general' with respect to the original decomposition (7), or requiring that various component-types be simultaneously non-empty, *et cetera*. We shall not pursue this here.

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2 Preliminaries

We begin by introducing a few mathematical preliminaries. In what follows, causal-geometric constructions, e.g. the causal future J^+, the causal past J^-, the chronological future I^+, the chronological past I^-, etc., will refer to the underlying flat Minkowski metric $\eta_{\mu\nu}$ and its induced topology on $\mathbb{R}^{1+1}$.[44]

2.1 Spacetime geometry of the maximal future development

We consider those spacetimes that, as in Theorems 1.1 and 1.11, are the maximal future developments of spherically symmetric initial data. In constructing the developments a local existence result is needed. We apply an easy generalization of a theorem of Choquét-Bruhat and Geroch [12], together with standard preservation of symmetry arguments, to obtain

Proposition 2.1. *Let $(\Sigma^{(3)}, \ldots)$ be a smooth spherically symmetric initial data set for the Einstein-Maxwell-Klein-Gordon system, where, topologically, $\Sigma^{(3)}$ is homeomorphic to either $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$. Then, there exists a unique smooth collection $(\mathcal{M}, g_{\mu\nu}, \phi, F_{\mu\nu})$ such that*

1. *$g_{\mu\nu}$, ϕ and $F_{\mu\nu}$ satisfy the Einstein-Maxwell-Klein-Gordon equations (2)–(6);*
2. *$(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic and $\Sigma^{(3)}$ is a Cauchy surface;*
3. *$(\mathcal{M}, g_{\mu\nu}, \phi, F_{\mu\nu})$ induces the initial data set $(\Sigma^{(3)}, \ldots)$; and,*
4. *any other collection $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}}, \widetilde{\phi}, \widetilde{F_{\mu\nu}})$ satisfying Properties 1–3 can be embedded into the given one.*

Moreover, $SO(3)$ acts smoothly by isometry on $\mathcal{M}$ and preserves $\phi\phi^\dagger$ and $F_{\mu\nu}$, and the quotient manifold $\mathcal{M}/SO(3)$ inherits the structure of a 2-dimensional, time-oriented Lorentzian manifold (possibly with boundary) that can be conformally embedded as a bounded subset of $(\mathbb{R}^{1+1}, \eta_{\mu\nu})$.

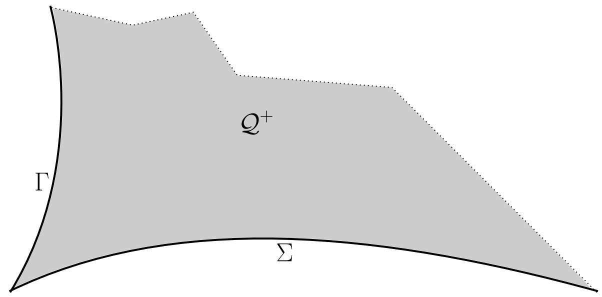
Let Γ denote the projection to $\mathcal{M}/SO(3)$ of the set of fixed points of the $SO(3)$-action on $\mathcal{M}$. Γ is then the (possibly empty, not necessarily connected) timelike boundary of $\mathcal{M}/SO(3)$, called the center of symmetry.

If $\Sigma^{(3)} \simeq \mathbb{R}^3$, then Γ is non-empty and has one connected component.
If $\Sigma^{(3)} \simeq \mathbb{S}^2 \times \mathbb{R}$, then Γ is empty.
If $\Sigma^{(3)} \simeq \mathbb{S}^3$, then Γ is non-empty and has two connected components.

Not immediately included in Proposition 2.1 is the case in which $\Sigma^{(3)} \simeq \mathbb{S}^2 \times \mathbb{S}^1$. In this case, however, we can apply the result to the universal cover of $\Sigma^{(3)}$, which has topology $\mathbb{S}^2 \times \mathbb{R}$.

Since our spacetime manifold $\mathcal{M}$ admits a spherically symmetric spacelike Cauchy hypersurface $\Sigma^{(3)}$, if $\pi : \mathcal{M} \to \mathcal{M}/SO(3)$ is the standard projection map, then $\Sigma = \pi(\Sigma^{(3)})$ is a connected spacelike curve 'downstairs' in $\mathcal{M}/SO(3)$, as $\Sigma^{(3)}$ is preserved under the $SO(3)$-action.

Let $\mathcal{Q}^+ = J^+(\Sigma^{(3)})/SO(3)$. For the asymptotically flat with one end initial data considered in Theorem 1.1, we have $\Sigma^{(3)} \simeq \mathbb{R}^3$. In which case, past-directed causal curves issuing



from a point $p \in \mathcal{Q}^+$ will terminate at a single point on $\Sigma \cup \Gamma$. Thus, Σ is the past (spacelike) boundary of $\mathcal{Q}^+$ as depicted below.

As conformal embeddings preserve causal structure, the standard double null co-ordinates (u, v) of the ambient $\mathbb{R}^{1+1}$ provide a global chart on $\mathcal{Q}^+$, and its metric can be written $g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\Omega^2\mathrm{d}u\mathrm{d}v$. In particular, the full metric on $\mathcal{M}$ is given by

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Omega^2\mathrm{d}u\mathrm{d}v + r^2 h$$

where $h = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ is the standard metric on $\mathbb{S}^2$ and $r : \mathcal{Q}^+ \to [0, \infty)$ is the area-radius function given by

$$r(p) = \sqrt{\frac{\mathrm{Area}(\pi^{-1}(p))}{4\pi}}.$$

The embedding into $\mathbb{R}^{1+1}$ shall be chosen, in the case that Γ is non-empty and connected, such that $\partial/\partial u$ points 'inward' towards Γ and $\partial/\partial v$ points 'outward' away from Γ.

2.2 Determined system

In order to work with a determined Einstein-Maxwell-Klein-Gordon system of equations, we must fix the inherent gauge freedom that arises.

2.2.1 Choice of gauge

The system of equations (2)–(6) is invariant under local $U(1)$ gauge transformations

$$\begin{aligned} \phi &\to e^{-\mathfrak{e}i\chi}\phi \\ A_\mu &\to A_\mu + \partial_\mu\chi, \end{aligned}$$

for any smooth real-valued function χ. Fixing a gauge amounts to fixing χ.

Let us recall that the electromagnetic field strength 2-form F can be expressed as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Although the 1-form A need *not* be spherically symmetric, the $SO(3)$ group action does, by assumption, preserve F. In particular, F can have only two non-vanishing components:

$$F = F_{uv}\mathrm{d}u \wedge \mathrm{d}v + F_{\theta\varphi}\mathrm{d}\theta \wedge \mathrm{d}\varphi.$$

Since every 2-form is proportional to the volume element on $\mathbb{S}^2$, it then follows from the fact that $\mathrm{d}F = 0$ that

$$F_{\theta\varphi} = Q_\mathrm{m}\sqrt{4\pi}\sin\theta \tag{15}$$

[44]We take the convention $p \in J^+(p) \cap J^-(p)$, but $p \notin I^+(p) \cup I^-(p)$.

for some constant Q_{m} such that[45] (cf. the Dirac quantization condition)

$$Q_{\mathrm{m}}\sqrt{4\pi} \in \frac{1}{2}\mathbb{Z}.$$

The quantity $Q_{\mathrm{m}}\sqrt{4\pi}$ is to be interpreted as the (topological) magnetic charge of $\mathcal{M}$.

Unless F is (de Rham) cohomologically trivial, there is no globally-defined 1-form A (on $\mathcal{M}$) such that $\mathrm{d}A = F$. Given F, we can define uniquely, however, a gauge on $\mathcal{M}\backslash\{\theta = \pi\}$ by imposing the following conditions:

$$\begin{aligned} A_u(u, v, \theta, \varphi)\big|_{\pi^{-1}(\Sigma\cup\Gamma)} &= 0 \\ A_v(u, v, \theta, \varphi) &= 0 \\ A_\theta(u, v, \theta, \varphi) &= 0 \\ A_\varphi(u, v, \theta = 0, \varphi) &= 0. \end{aligned}$$

Note that

$$A_u \mathrm{d}u$$

is a globally-defined 1-form on $\mathcal{Q}^+$. In this choice of gauge, we have that

$$\partial_\theta A_\varphi = F_{\theta\varphi} = Q_{\mathrm{m}}\sqrt{4\pi}\sin\theta,$$

and hence

$$A_\varphi = Q_{\mathrm{m}}\sqrt{4\pi}\left(1 - \cos\theta\right).$$

In spherical symmetry, we have the requirement that $\mathrm{D}_\theta\phi = \mathrm{D}_\varphi\phi = 0$. Since

$$\mathrm{D}_\theta\phi = \partial_\theta\phi = 0,$$

we note that ϕ is independent of the co-ordinate θ. On the other hand, since

$$\mathrm{D}_\varphi\phi = \partial_\varphi\phi + \mathfrak{e}\mathrm{i}A_\varphi\phi = 0,$$

we have, upon integration, that

$$\phi(u, v, \varphi) = \exp\left(-\mathfrak{e}\mathrm{i}Q_{\mathrm{m}}\sqrt{4\pi}\left(1 - \cos\theta\right)\varphi\right)\phi(u, v, \varphi = 0).$$

Thus, ϕ on each group orbit is determined by the value of ϕ at a single point on the 2-sphere (modulo a phase). In particular, we note that $\phi\phi^\dagger$ is spherically symmetric as

$$\phi\phi^\dagger(u, v, \varphi) = \phi\phi^\dagger(u, v, \varphi = 0),$$

i.e. $\phi\phi^\dagger$ depends only on u and v.

2.2.2 Energy-momentum tensor

Let us note that the energy-momentum tensor (3)–(4) immediately gives

$$\begin{aligned} T_{uu} &= \mathrm{D}_u\phi\left(\mathrm{D}_u\phi\right)^\dagger \\ T_{vv} &= \partial_v\phi\left(\partial_v\phi\right)^\dagger. \end{aligned}$$

To compute the other components of the energy-momentum tensor, let us consider the scalar invariant $F_{\mu\nu}F^{\mu\nu}$. We note that in spherical symmetry,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \\ &= 2g^{\theta\theta}g^{\varphi\varphi}\left(F_{\theta\varphi}\right)^2 - 2\left(g^{uv}\right)^2\left(F_{uv}\right)^2 \\ &= \frac{8\pi}{r^4}Q_{\mathrm{m}}^2 - 8\Omega^{-4}\left(F_{uv}\right)^2. \end{aligned}$$

[45] In spherical symmetry, however, $Q_{\mathrm{m}}\sqrt{4\pi}$ need not be discrete (cf. §2.2.5).

Let us define the quantity Q_{e} (called the electric charge) by

$$\sqrt{\pi}Q_{\mathrm{e}} = r^2\Omega^{-2}F_{uv}. \tag{16}$$

Noting the identity

$$F_{\mu\nu}F^{\mu\nu} = -\frac{8\pi}{r^4}\left(Q_{\mathrm{e}}^2 - Q_{\mathrm{m}}^2\right),$$

the (u,v)-component of (3)–(4) then reads

$$T_{uv} = \frac{1}{4}\Omega^2\left(\mathfrak{m}^2\phi\phi^\dagger + \frac{Q_{\mathrm{e}}^2 + Q_{\mathrm{m}}^2}{r^4}\right). \tag{17}$$

Lastly, we note that the spherical components of the energy-momentum tensor are given by

$$T_{\varphi\varphi} = T_{\theta\theta}\sin^2\theta = r^2\Omega^{-2}\left(\mathrm{D}_u\phi\,(\partial_v\phi)^\dagger + \partial_v\phi\,(\mathrm{D}_u\phi)^\dagger - \frac{1}{2}\mathfrak{m}^2\Omega^2\phi\phi^\dagger\right) + \frac{Q_{\mathrm{e}}^2 + Q_{\mathrm{m}}^2}{2r^2}.$$

Since $T_{uu} \geq 0$ and $T_{vv} \geq 0$, the Einstein-Maxwell-Klein-Gordon system obeys the null energy condition. If, in addition, $\mathfrak{m}^2 \geq 0$, then $T_{uv} \geq 0$ and thus the model obeys the dominant energy condition. We shall assume $\mathfrak{m}^2 \geq 0$ henceforth.[46]

2.2.3 Magnetic charge

For spacetimes with one asymptotically flat end, as in the context of Theorem 1.1, we note that the initial Cauchy hypersurface $\Sigma^{(3)}$ in $\mathcal{M}$ has topology $\mathbb{R}^3$. In particular, Q_{m}^2 necessarily vanishes for these spacetimes. Because, however, we do not make any assumptions on the topology of initial data in Theorem 1.11, we will *not* assume in the sequel that Q_{m}^2 vanishes.

2.2.4 Reduced system of equations

The spherically symmetric Einstein-Maxwell-Klein-Gordon system reduces (in the gauge of §2.2.1) to the following equations on $\mathcal{Q}^+$ for C^2-functions (r,Ω,ϕ,A_u):

$$r\partial_v\partial_u r = -\frac{1}{4}\Omega^2 - \partial_v r\partial_u r + \mathfrak{m}^2\pi r^2\Omega^2\phi\phi^\dagger + \pi\Omega^2 r^{-2}Q^2 \tag{18}$$

$$r^2\partial_u\partial_v\log\Omega^2 = -2\pi r^2\left(\mathrm{D}_u\phi\,(\partial_v\phi)^\dagger + \partial_v\phi\,(\mathrm{D}_u\phi)^\dagger\right) - 2\pi\Omega^2 r^{-2}Q^2 + \frac{1}{4}\Omega^2 + \partial_u r\partial_v r \tag{19}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r\Omega^{-2}\mathrm{D}_u\phi\,(\mathrm{D}_u\phi)^\dagger \tag{20}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r\Omega^{-2}\partial_v\phi\,(\partial_v\phi)^\dagger \tag{21}$$

$$\partial_u Q_{\mathrm{e}} = \mathfrak{e}\mathrm{i}\sqrt{\pi}r^2\left(\phi\,(\mathrm{D}_u\phi)^\dagger - \phi^\dagger\mathrm{D}_u\phi\right) \tag{22}$$

$$\partial_v Q_{\mathrm{e}} = -\mathfrak{e}\mathrm{i}\sqrt{\pi}r^2\left(\phi\,(\partial_v\phi)^\dagger - \phi^\dagger\partial_v\phi\right) \tag{23}$$

$$\partial_u\partial_v\phi + r^{-1}(\partial_u r\partial_v\phi + \partial_v r\partial_u\phi) + \mathfrak{e}\mathrm{i}\Psi(A) = -\frac{1}{4}\mathfrak{m}^2\Omega^2\phi \tag{24}$$

$$\Psi(A) = A_u\left(\phi\partial_v\log r + \partial_v\phi\right) - \frac{1}{2}\sqrt{\pi}\Omega^2 r^{-2}Q_{\mathrm{e}}\phi \tag{25}$$

$$Q^2 = Q_{\mathrm{e}}^2 + Q_{\mathrm{m}}^2 \tag{26}$$

$$\partial_v A_u = -\sqrt{\pi}\Omega^2 r^{-2}Q_{\mathrm{e}}. \tag{27}$$

[46] In the case of the Einstein-Klein-Gordon system ($\mathfrak{e} = F_{\mu\nu} = 0$), the assumption that $\mathfrak{m}^2 \geq 0$ can be dropped in the proof of Theorem 1.11. See footnote 50.

2.2.5 Quantization condition

Let us note that while the restriction $\mathfrak{e} \in \mathbb{Z}$ and $\sqrt{4\pi}Q_{\mathrm{m}} \in \frac{1}{2}\mathbb{Z}$ appears in the set-up of §1.1.1 and §2.2.1, in the reduced spherically symmetric system (18)–(27) we can take these constants to be, in fact, arbitrary. In particular, we can take the continuous limit $\mathfrak{e} \to 0$ (and the continuous limit $\mathfrak{m}^2 \to 0$) in the spherically symmetric Einstein-Maxwell-Klein-Gordon system to obtain the special models of Christodoulou and Dafermos (cf. §1.4).

3 Rudimentary boundary characterization

Let $\mathcal{Q}^+$ be as in Theorem 1.1. The maximal future development $\mathcal{Q}^+$ will have boundary $\mathcal{B}^+ = \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$, of which we aim to now characterize.[47]

The following description of $\mathcal{B}^+$ holds very generally for globally hyperbolic spherically symmetric spacetimes with one asymptotically flat end where the matter model obeys the null energy condition. The boundary $\mathcal{B}^+$ can be decomposed into two components: the boundary 'emanating from spacelike infinity' and the boundary 'emanating from first singularities'.

We begin with a few preliminary results.

3.1 Nowhere anti-trapped initial data

The statement of Theorem 1.1 assumes that initial data are prescribed such that no anti-trapped regions are present, i.e.

$$\partial_u r < 0$$

along Σ. We thus assume this throughout §3. Data satisfying this property, motivated by Christodoulou in [18], preclude anti-trapped regions from forming in their future development; that is, anti-trapped regions are non-evolutionary. This is given in

Proposition 3.1 (Christodoulou)**.** *In the notation of Theorem 1.1, if $\partial_u r < 0$ along Σ, then $\partial_u r < 0$ everywhere in $\mathcal{Q}^+$.*

Proof. By global hyperbolicity, the past-directed null segment of constant-v issuing from $(u, v) \in \mathcal{Q}^+$ will terminate on Σ. In particular, there exists $u' \leq u$ such that $\partial_u r(u', v) < 0$, by assumption. The result follows by integrating (20) along $[u', u] \times \{v\}$. $\square$

We emphasize that Proposition 3.1 only relies on the null energy condition.

3.2 Spacetime volume

Let us define the sets

$$\begin{aligned} \mathcal{U} &= \{u : \exists v \;\text{ s.t. }\; (u, v) \in \mathcal{Q}^+\} \\ \mathcal{V} &= \{v : \exists u \;\text{ s.t. }\; (u, v) \in \mathcal{Q}^+\}. \end{aligned}$$

Since $\mathcal{Q}^+$ is a bounded subset of $\mathbb{R}^{1+1}$, we have that

$$\begin{aligned} U &= \sup \mathcal{U} < \infty \\ V &= \sup \mathcal{V} < \infty, \end{aligned}$$

and

$$\begin{aligned} U_0 &= \inf \mathcal{U} > -\infty \\ V_0 &= \inf \mathcal{V} > -\infty. \end{aligned}$$

[47] This boundary is induced by the conformal embedding of $\mathcal{Q}^+$ into $\mathbb{R}^{1+1}$ and is *not* the boundary in the sense of a manifold-with-boundary, which is $\Sigma \cup \Gamma$.

For any $u' \le U$ and $v' < V$ such that $(u', v') \in \mathcal{Q}^+$, it follows by Proposition 3.1 and monotonicity (20) that the spacetime volume of the region

$$\mathcal{Q}^+(u', v') = \mathcal{Q}^+ \cap \{u \le u'\} \cap \{v \le v'\}$$

is given by

$$\begin{aligned}
\int_{\mathcal{Q}^+(u',v')} 2 \text{ dVol} &= \int_{\mathcal{Q}^+\cap\{v\le v'\}} \int_{\mathcal{Q}^+\cap\{u\le u'\}} (-\Omega^{-2}\partial_u r)^{-1}(-\partial_u r) \text{ d}u\text{d}v \\
&\le \sup_{\Sigma\cap\{v\le v'\}} \Omega^2 |\partial_u r| \int_{\mathcal{Q}^+\cap\{v\le v'\}} \int_{\mathcal{Q}^+\cap\{u\le u'\}} |\partial_u r| \text{ d}u\text{d}v \\
&\le \sup_{\Sigma\cap\{v\le v'\}} r\Omega^2 |\partial_u r| \int_{\mathcal{Q}^+\cap\{v\le v'\}} \text{d}v \\
&\le (V - V_0) \left(\sup_{\Sigma\cap\{v\le v'\}} r\Omega^2 |\partial_u r| \right) < \infty.
\end{aligned}$$

We have thus established, in particular,

Proposition 3.2. *If $p \in \mathcal{Q}^+$, then the region $J^-(p) \cap \mathcal{Q}^+$ has finite spacetime volume.*

In fact, we note that the domain of dependence of any compact subset of $\Sigma^{(3)}$ has finite spacetime volume 'upstairs' as well. To see this, we simply compute that

$$\int_{\mathcal{M}} 2 \text{ dVol}_{\mathcal{M}} = \int_{\mathcal{M}} r^2 \Omega^2 \sin\theta \text{ d}u\text{d}v\text{d}\theta\text{d}\varphi = 4\pi \int_{\mathcal{Q}^+} r^2 \text{ dVol}_{\mathcal{Q}^+}.$$

Since $\partial_u r < 0$ in $\mathcal{Q}^+$, r is uniformly bounded in any compact subset of $\mathcal{Q}^+$ as

$$\sup_{\mathcal{Q}^+\cap\{v\le v'\}} r \le \sup_{\Sigma\cap\{v\le v'\}} r < \infty.$$

This establishes the claim.

3.3 Boundary 'emanating from spacelike infinity'

The initial data curve Σ acquires a unique limit point $i^0 \in \mathcal{B}^+$ called *spacelike infinity*.

Recall the notation of §3.2. Let $\mathcal{U}_\infty$ denote the set of all u given by

$$\mathcal{U}_\infty = \left\{ u : \sup_{\mathcal{V}} r(u, v) = \infty \right\},$$

which may be, *a priori*, empty (even if $r \to \infty$ along Σ). For each $u \in \mathcal{U}_\infty$, there exists a unique $v_\infty(u)$ such that

$$(u, v_\infty(u)) \in \mathcal{B}^+.$$

Define *future null infinity* $\mathcal{I}^+$ by

$$\mathcal{I}^+ = \bigcup_{u\in\mathcal{U}_\infty} (u, v_\infty(u)).$$

For data of compact support along Σ, an application of Birkhoff's theorem and the domain of dependence property will ensure that $\mathcal{I}^+$ is non-empty, allowing us then to appeal to

Proposition 3.3. *If non-empty, $\mathcal{I}^+$ is a connected ingoing null segment with past limit point i^0.*

Proof. Let $i^0 = (U, V)$. Since, by Proposition 3.1, $\partial_u r < 0$ in $\mathcal{Q}^+$, for all $v_0 < V$ we have

$$\sup_{\mathcal{Q}^+ \cap \{v \le v_0\}} r \le \sup_{\Sigma \cap \{v \le v_0\}} r.$$

Thus, $\mathcal{I}^+ \cap \{v = v_0\} = \emptyset$. In particular, $\mathcal{I}^+ \subset \{v = V\}$.

Consider $(u_0, V) \in \mathcal{I}^+$, and let $u < u_0$ be such that $u \in \mathcal{U}$. Using, again, the fact that $\partial_u r < 0$ in $\mathcal{Q}^+$, we obtain

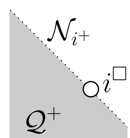
$$\lim_{v \to V} r(u, v) \ge \lim_{v \to V} r(u_0, v) = \infty.$$

That is, $(u, V) \in \mathcal{I}^+$. $\square$

Alternatively, the non-emptiness of $\mathcal{I}^+$ easily follows if the fall-off rate of data at space-like infinity is suitably tame.[48] An appropriate notion, therefore, of 'asymptotically flat' in Theorem 1.1 can, *ab initio*, ensure that $\mathcal{I}^+$ is non-empty. We will assume that we have built in the requirement $\mathcal{I}^+ \neq \emptyset$ into our definition of asymptotically flat in Theorem 1.1.

We define $i^\square \in \mathcal{B}^+$ as the future limit point of $\mathcal{I}^+$, noting, *a priori*, we could have $i^\square \in \mathcal{I}^+$.

By causality, we have a (possibly empty) half-open null segment[49] $\mathcal{N}_{i^+}$ emanating from (but not including) $i^\square$.



3.4 Boundary 'emanating from first singularities'

We now introduce the notion of a point $p \in \overline{\mathcal{Q}^+} \setminus \overline{\mathcal{I}^+}$ being a 'first singularity'.

Definition 4. Let $p \in \overline{\mathcal{Q}^+}$. The causal set $J^-(p) \cap \mathcal{Q}^+ \subset \mathcal{Q}^+$ is said to be *compactly generated* if there exists a compact subset $X \subset \mathcal{Q}^+$ such that

$$J^-(p) \subset \pi\left(D^+_{\mathcal{M}}(\pi^{-1}(X))\right) \cup J^-(X).$$

If $p \in \overline{\mathcal{Q}^+} \setminus \overline{\Gamma}$, then for $J^-(p) \cap \mathcal{Q}^+$ to be compactly generated means that there exists a causal rectangle

$$\mathcal{D} = \left(J^-(p) \cap J^+(q)\right) \setminus \{p\} \subset \mathcal{Q}^+$$

for some $q \in (I^-(q) \cap \mathcal{Q}^+) \setminus \{p\}$.

If $p \in \Gamma$, $\overline{J^-(p) \cap \mathcal{Q}^+}$ is always compactly generated. Let b_Γ denote the unique future limit point of Γ in $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$. If $p = b_\Gamma$, then $J^-(p) \cap \mathcal{Q}^+$ is compactly generated as long as $b_\Gamma \notin \mathcal{N}_{i^+} \cup i^\square$.

Definition 5. $p \in \mathcal{B}^+$ is said to be a *first singularity* if the causal set $J^-(p) \cap \mathcal{Q}^+$ is compactly generated and if any compactly generated proper causal subset of $J^-(p) \cap \mathcal{Q}^+$ is of the form $J^-(q)$ for a $q \in \mathcal{Q}^+$.

Definition 6. Let $\mathcal{B}^+_1 \subset \mathcal{B}^+$ be the set of all first singularities. A first singularity $p \in \mathcal{B}^+_1 \setminus b_\Gamma$ will be called *non-central.*

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[48] We will not discuss the weakest possible notion of an admissible data set here, but the reader may wish to consider the issue further. It is certainly sufficient for our purpose to consider the case where ϕ is compactly supported, in which case, by an extension of Birkhoff's theorem, the geometry of Σ coincides with Reissner-Nordström outside the support of ϕ.

[49] We choose the notation $\mathcal{N}_{i^+}$ as this set can be non-empty only if $i^\square = i^+$ (cf. Statement III of Theorem 1.1).

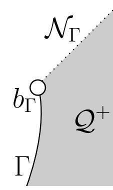
If $\mathcal{B}_1^+ = \emptyset$, then $\mathcal{B}^+$ is given by

$$\mathcal{B}^+ = b_\Gamma \cup \mathcal{N}_{i^+} \cup i^\square \cup \mathcal{I}^+ \cup i^0,$$

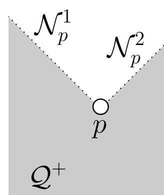
where b_Γ and the future endpoint of $\mathcal{N}_{i^+} \cup i^\square$ coincide.

We note that if $\mathcal{B}_1^+ \neq \emptyset$, then $b_\Gamma \in \mathcal{B}_1^+$. We then define the boundary 'emanating from first singularities' as the union of

1. b_Γ and a (possibly empty) half-open null segment $\mathcal{N}_\Gamma$ emanating from (but not including) b_Γ; and,



2. the set $\mathcal{B}_1^+ \backslash b_\Gamma$ and for every $p \in \mathcal{B}_1^+ \backslash b_\Gamma$, two (possibly empty) half-open null segments $\mathcal{N}_p^j$ emanating from (but not including) p.



3.5 Summary of preliminary boundary decomposition

We summarize the boundary decomposition thus far in

Proposition 3.4. *Let $(\mathcal{M} = \mathcal{Q}^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ denote the maximal future development of initial data as in Theorem 1.1. The Penrose diagram has boundary $\mathcal{B}^+$ admitting the (not necessarily disjoint) decomposition*

$$\mathcal{B}^+ = b_\Gamma \cup \mathcal{N}_\Gamma \cup \left(\bigcup_{p \in \mathcal{B}_1^+ \backslash b_\Gamma} \{p\} \cup \mathcal{N}_p^1 \cup \mathcal{N}_p^2 \right) \cup \mathcal{N}_{i^+} \cup i^\square \cup \mathcal{I}^+ \cup i^0,$$

where the sets are as described previously. Moreover, the following hold:

1. *If $\mathcal{B}_1^+ = \emptyset$, then $\mathcal{N}_\Gamma = \emptyset$.*

2. *If $\mathcal{B}_1^+ \backslash b_\Gamma = \emptyset$, then the future endpoint of $b_\Gamma \cup \mathcal{N}_\Gamma$ coincides with the future endpoint of $\mathcal{N}_{i^+} \cup i^\square$.*

4 The generalized extension principle

In this section, we shall briefly shift gears and prove the generalized extension principle of Theorem 1.11. Here, $\mathcal{Q}^+$ will denote the maximal development of the more general initial data as in the statement of Theorem 1.11. In §5, we will return to considering the development of initial data as in the statement of Theorem 1.1.

4.1 Another local existence result

The extension criterion, given in the next subsection, relies on an auxiliary (to Proposition 2.1) local existence statement, which is stated directly at the level of the spherically symmetric reduction. Formulated as a double characteristic initial value problem, the following result is suitable for our purpose.

Proposition 4.1. *Let $k \geq 0$ and consider a set $X = [0, d] \times \{0\} \cup \{0\} \times [0, d]$. On X, let r be a positive function that is C^{k+2} and functions Ω, ϕ and A_u that are C^{k+1}. Suppose that equations (20) and (21) hold initially on $[0, d] \times \{0\}$ and $\{0\} \times [0, d]$, respectively. Let $|\cdot|_{n,u}$ denote the $C^n(u)$-norm on $[0, d] \times \{0\}$; similarly, let $|\cdot|_{n,v}$ denote the $C^n(v)$-norm on $\{0\} \times [0, d]$. Define*

$$N = \sup\{|\Omega|_{1,u}, |\Omega|_{1,v}|\Omega^{-1}|_0, |r|_{2,u}, |r|_{2,v}, |r^{-1}|_0, |\phi|_{1,u}, |\phi|_{1,v}, |A_u|_{1,u}, |A_u|_{1,v}\}.$$

There exists $\delta > 0$, depending only on N, a C^{k+2}-function (unique amongst C^2-functions) r and C^{k+1}-functions (unique amongst C^1-functions) Ω, ϕ and A_u satisfying equations (18)–(27) in $[0, \delta^*] \times [0, \delta^*]$, where $\delta^* = \min\{d, \delta\}$, such that the restriction of these functions to X is as prescribed.

The above result is quite general in the sense that it does not depend on the structure of the non-linear terms in (2)–(6). The proof is omitted, but can be obtained via standard arguments (cf. the appendix of [30]).

4.2 An extension criterion

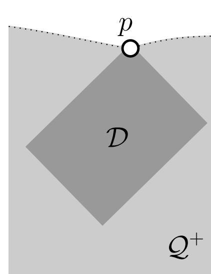
Let $p = (U, V) \in \overline{\mathcal{Q}^+}$, $q = (U', V') \in (I^-(p) \cap \mathcal{Q}^+) \setminus \{p\}$ and

$$\mathcal{D} = \left(J^+(q) \cap J^-(p)\right) \setminus \{p\} \subset \mathcal{Q}^+$$

be as in the statement of Theorem 1.11. Then, the compact set

$$X = [U', U] \times \{V'\} \cup \{U'\} \times [V', V] \tag{28}$$

satisfies $X \subset \mathcal{Q}^+$.



Given a subset $Y \subset \mathcal{Q}^+$, we define a 'norm' $N : Y \to [0, \infty]$ given by

$$N(Y) = \sup\{|\Omega|_1, |\Omega^{-1}|_0, |r|_2, |r^{-1}|_0, |\phi|_1, |A_u|_1\},$$

where $|f|_n$ denotes the restriction of the C^n-norm on $\mathcal{Q}^+$ to Y.

In view of Proposition 4.1, we formulate an extension criterion in

Proposition 4.2. *Let $p = (U, V) \in \overline{\mathcal{Q}^+}$ and $q = (U', V') \in (I^-(p) \cap \mathcal{Q}^+) \setminus \{p\}$ be such that*

$$\mathcal{D} = \left(J^+(q) \cap J^-(p)\right) \setminus \{p\} \subset \mathcal{Q}^+.$$

Then,

$$N(\mathcal{D}) = \infty.$$

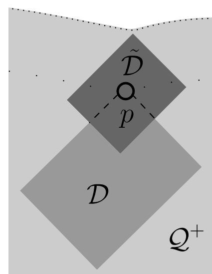
Proof. We prove the contrapositive. Let $X \subset \mathcal{Q}^+$ be the compact set as in (28) with $N = N(\mathcal{D}) < \infty$. Corresponding to the value N, let $\delta > 0$ be given as in Proposition 4.1. Consider the point $(U - \frac{1}{2}\delta, V - \frac{1}{2}\delta)$. Without loss of generality, we can assume that this point is in $\mathcal{Q}^+$. Translate the co-ordinates so that this point is called $(0, 0)$. Since $\mathcal{Q}^+$ is, by definition, an open set, there is by continuity a $\delta^* \in (\frac{1}{2}\delta, \delta)$ such that

$$X^* = \{0\} \times [0, \delta^*] \cup [0, \delta^*] \times \{0\} \subset \mathcal{Q}^+.$$

Moreover, the assumptions of Proposition 4.1 hold on this set X^*. Hence, there exists a unique solution in

$$\tilde{\mathcal{D}} = [0, \delta^*] \times [0, \delta^*]$$

that coincides with the previous solution on $\mathcal{D} \cap \tilde{\mathcal{D}}$ by uniqueness.



As $\tilde{\mathcal{D}} \cup \mathcal{Q}^+$ is the quotient of a development of initial data, we must have, by maximality of $\mathcal{Q}^+$, that $\tilde{\mathcal{D}} \cup \mathcal{Q}^+ \subset \mathcal{Q}^+$. Thus, in particular, $p \in \mathcal{Q}^+$. □

4.3 Proof of Theorem 1.11: generalized extension principle

Define

$$W = \int_{V'}^{V} \int_{U'}^{U} \Omega^2 \, \mathrm{d}u\mathrm{d}v < \infty \tag{29}$$

and let r_0 and R be constants such that

$$0 < r_0 \leq r(u, v) \leq R < \infty$$

for all $(u, v) \in \mathcal{D}$.

Let us recall (16) in which

$$\sqrt{\pi} Q_{\mathrm{e}} = r^2 \Omega^{-2} F_{uv}.$$

We introduce further the notation

$$\begin{aligned}
\lambda &= \partial_v r \\
\nu &= \partial_u r \\
\theta_A &= r \partial_v \phi \\
\zeta_A &= r \mathrm{D}_u \phi.
\end{aligned}$$

By compactness, and the regularity of a solution as given in Theorem 1.11, we have the following uniform bounds on $X = [U', U] \times \{V'\} \cup \{U'\} \times [V', V]$:

$$|r\nu| \leq N$$

$$|r\lambda| \leq \Lambda$$

$$|r\phi| \leq \Phi$$

$$|\theta_A| \leq \Theta$$
$$|\zeta_A| \leq Z$$
$$|Q_{\mathrm{e}}| \leq B$$
$$|A_u| \leq A$$
$$|\partial_u \nu|, |\partial_v \lambda|, |\partial_u \Omega|, |\partial_v \Omega|, |\log \Omega^2| \leq H.$$

In view of the extension criterion in Proposition 4.2, it suffices to show that similar uniform bounds hold in $\mathcal{D}$ in order to prove Theorem 1.11.

4.3.1 *a priori* integral estimates

Recalling the notation of (17), we begin by integrating (18) in u and v to obtain

$$\begin{aligned} \int_{V'}^{V} \int_{U'}^{U} 4\pi r^2 T_{uv} \, \mathrm{d}u\mathrm{d}v &= \int_{V'}^{V} \int_{U'}^{U} \partial_u (r\lambda) \, \mathrm{d}u\mathrm{d}v + \frac{1}{4} \int_{V'}^{V} \int_{U'}^{U} \Omega^2 \, \mathrm{d}u\mathrm{d}v \\ &\leq 4R(R - r_0) + \frac{1}{4} W =: C_0, \end{aligned} \tag{30}$$

where we have made use of the fact that λ will change sign at most once along a ray of constant u. This fact follows from monotonicity given in (21).

Using (30), we obtain *a priori* spacetime integral bounds on the matter fields. In particular, we have[50]

$$\pi \mathfrak{m}^2 \int_{V'}^{V} \int_{U'}^{U} \Omega^2 |r\phi|^2 \, \mathrm{d}u\mathrm{d}v \leq C_0 \tag{31a}$$
$$\pi \int_{V'}^{V} \int_{U'}^{U} \Omega^2 r^{-2} Q^2 \, \mathrm{d}u\mathrm{d}v \leq C_0. \tag{31b}$$

Note that (31b) holds with Q^2 replaced by Q_{e}^2 on account of (26).

We now integrate (18) in u to yield the pointwise estimate

$$\sup_{U' \leq u \leq U} |r\lambda| \leq \Lambda + \frac{1}{4} \int_{U'}^{U} \Omega^2 \, \mathrm{d}u + \int_{U'}^{U} 4\pi r^2 T_{uv} \, \mathrm{d}u. \tag{32}$$

Upon integration in v, we then have

$$\int_{V'}^{V} \sup_{U' \leq u \leq U} |r\lambda| \, \mathrm{d}v \leq \Lambda(V - V') + \frac{1}{4} W + C_0 =: C_1. \tag{33}$$

Similarly, we obtain the estimate

$$\int_{U'}^{U} \sup_{V' \leq v \leq V} |r\nu| \, \mathrm{d}u \leq N(U - U') + \frac{1}{4} W + C_0 =: C_2. \tag{34}$$

A Cauchy-Schwarz inequality and (31b) gives

$$\begin{aligned} \int_{U'}^{U} \int_{V'}^{V} |F_{uv}| \, \mathrm{d}v\mathrm{d}u &\leq \left(\int_{U'}^{U} \int_{V'}^{V} \pi \Omega^2 r^{-2} Q_{\mathrm{e}}^2 \, \mathrm{d}v\mathrm{d}u \right)^{\frac{1}{2}} \left(\int_{U'}^{U} \int_{V'}^{V} \Omega^2 r^{-2} \, \mathrm{d}v\mathrm{d}u \right)^{\frac{1}{2}} \\ &\leq \sqrt{C_0} r_0^{-1} \sqrt{W}. \end{aligned} \tag{35}$$

[50] It is here, and only here, where we use the dominant energy condition in the proof of Theorem 1.11, i.e. the assumption that $\mathfrak{m}^2 \geq 0$. In the case of the Einstein-Klein-Gordon system ($\mathfrak{e} = F_{\mu\nu} = 0$), we note that (31) holds an account of T_{uv} having only one component (since $Q^2 = 0$).

We immediately then obtain

$$\begin{aligned}\int_{U'}^{U}\sup_{V'\leq v\leq V}|A_u|\ \mathrm{d}u &\leq A(U-U')+\int_{U'}^{U}\int_{V'}^{V}|\partial_v A_u|\ \mathrm{d}v\mathrm{d}u \\ &\leq A(U-U')+\sqrt{C_0}r_0^{-1}\sqrt{W}=:C_3. \end{aligned} \tag{36}$$

4.3.2 Uniform bound on $r\phi$

Given $\epsilon>0$, partition the region of spacetime $\mathcal{D}$ into smaller subregions $\mathcal{D}_{jk}$ given by

$$\mathcal{D}_{jk}=[u_j,u_{j+1}]\times[v_k,v_{k+1}]\cap\mathcal{D},\quad j,k=0,\ldots,I$$

with $u_0=U'$, $v_0=V'$, $u_{I+1}=U$ and $v_{I+1}=V$ such that

$$\begin{aligned}\int_{v_k}^{v_{k+1}}\int_{u_j}^{u_{j+1}}\Omega^2\ \mathrm{d}u\mathrm{d}v &< \epsilon \\ \int_{v_k}^{v_{k+1}}\sup_{u_j\leq u\leq u_{j+1}}|r\lambda|\ \mathrm{d}v &< \epsilon \\ \int_{u_j}^{u_{j+1}}\sup_{v_k\leq v\leq v_{k+1}}|A_u|\ \mathrm{d}u &< \epsilon\end{aligned}$$

for all j and k. This is clearly possible since we have shown that each quantity is *uniformly* bounded in $\mathcal{D}$. Note that the cardinality of I for the partition depends on ϵ.

Let $(u_*,v_*)\in\mathcal{D}_{jk}$. Now re-write the wave equation (24) to read

$$\partial_u\partial_v(r\phi)=\phi\partial_u\lambda-\mathfrak{e}ir\Psi(A)-\frac{1}{4}\mathfrak{m}^2\Omega^2 r\phi.$$

Define

$$P_{jk}=\sup_{\mathcal{D}_{jk}}|r\phi|.$$

From (33) it follows that

$$\left|\int_{v_k}^{v_*}\int_{u_j}^{u_*}\phi\partial_u\lambda\ \mathrm{d}u\mathrm{d}v\right|\leq 2r_0^{-2}P_{jk}\int_{v_k}^{v_*}\sup_{u_j\leq u\leq u_*}|r\lambda|\ \mathrm{d}v\leq 2r_0^{-2}P_{jk}\epsilon. \tag{37}$$

From (36) we have

$$\begin{aligned}\left|\int_{v_k}^{v_*}\int_{u_j}^{u_*}A_u\phi\lambda\ \mathrm{d}u\mathrm{d}v\right| &\leq r_0^{-2}P_{jk}\int_{v_k}^{v_*}\sup_{u_j\leq u\leq u_*}|r\lambda|\ \mathrm{d}v\int_{u_j}^{u_*}\sup_{v_k\leq v\leq v_*}|A_u|\ \mathrm{d}u \\ &\leq r_0^{-2}P_{jk}\epsilon^2.\end{aligned}$$

Integrating by parts, we note that

$$\begin{aligned}\left|\int_{v_k}^{v_*}A_u\partial_v\phi\ \mathrm{d}v\right| &= \left|\phi A_u(u_*,v_*)-\phi A_u(u_*,v_k)+\int_{v_k}^{v_*}\phi\partial_v A_u\ \mathrm{d}v\right| \\ &\leq r_0^{-1}P_{jk}\left(2\sup_{v_k\leq v\leq v_*}|A_u|+\int_{v_k}^{v_*}|F_{uv}|\ \mathrm{d}v\right),\end{aligned}$$

so that (35) and (36) yield

$$\left|\int_{u_j}^{u_*}\int_{v_k}^{v_*}A_u r\partial_v\phi\ \mathrm{d}v\mathrm{d}u\right|\leq Rr_0^{-1}P_{jk}\left(2\epsilon+\sqrt{C_0}r_0^{-1}\sqrt{\epsilon}\right). \tag{38}$$

Similarly, inequality (35) implies that

$$\left|\int_{v_k}^{v_*}\int_{u_j}^{u_*}\frac{1}{2}\phi F_{uv}\ \mathrm{d}u\mathrm{d}v\right|\leq\frac{1}{2}P_{jk}r_0^{-2}\sqrt{C_0}\sqrt{\epsilon}. \tag{39}$$

Lastly, we note that

$$\left|\int_{v_k}^{v_*}\int_{u_j}^{u_*}\frac{1}{4}\mathfrak{m}^2\Omega^2 r\phi\ \mathrm{d}u\mathrm{d}v\right|\leq\frac{1}{4}\mathfrak{m}^2 P_{jk}\epsilon. \tag{40}$$

Upon integration of the wave equation, it then follows from (37)–(40) that there are constants C_4 and C_5, independent of the choice of ϵ and partition, such that

$$P_{jk}\leq C_4+C_5P_{jk}\epsilon.$$

Thus, for a sufficiently small ϵ, i.e. for a sufficiently fine partition of $\mathcal{D}$, there exists a positive constant $C_{jk}<\infty$ such that

$$P_{jk}\leq C_{jk}.$$

In particular, since

$$\sup_{\mathcal{D}}|r\phi|\leq\max_{j,k}P_{jk}<\infty, \tag{41}$$

we obtain a uniform bound on the scalar field.

4.3.3 Uniform bound on Ω^2

Consider the evolution equation (19). We recall (15) and (26) that the magnetic charge $Q_{\mathrm{m}}\in\mathbb{R}$. To uniformly bound Ω^2, it suffices, given *a priori* estimates (31b), (33) and (34), to bound

$$\int_{U'}^{u_*}\int_{V'}^{v_*}\mathrm{D}_u\phi\left(\partial_v\phi\right)^\dagger+\partial_v\phi\left(\mathrm{D}_u\phi\right)^\dagger\ \mathrm{d}v\mathrm{d}u \tag{42}$$

uniformly. Integrating by parts, we note that

$$\int_{V'}^{v_*}\mathrm{D}_u\phi\left(\partial_v\phi\right)^\dagger\ \mathrm{d}v=\phi^\dagger\mathrm{D}_u\phi(u,v_*)-\phi^\dagger\mathrm{D}_u\phi(u,V')-\int_{V'}^{v_*}\phi^\dagger\partial_v\mathrm{D}_u\phi\ \mathrm{d}v.$$

Similarly, we have

$$\int_{V'}^{v_*}\left(\mathrm{D}_u\phi\right)^\dagger\partial_v\phi\ \mathrm{d}v=\phi\left(\mathrm{D}_u\phi\right)^\dagger(u,v_*)-\phi\left(\mathrm{D}_u\phi\right)^\dagger(u,V')-\int_{V'}^{v_*}\phi\partial_v\left(\mathrm{D}_u\phi\right)^\dagger\ \mathrm{d}v.$$

Since

$$\begin{aligned}\phi\left(\mathrm{D}_u\phi\right)^\dagger&=\phi\left(\partial_u\phi\right)^\dagger-\mathfrak{e}\mathrm{i}A_u|\phi|^2\\ \phi^\dagger\mathrm{D}_u\phi&=\phi^\dagger\partial_u\phi+\mathfrak{e}\mathrm{i}A_u|\phi|^2,\end{aligned}$$

it follows that

$$\phi^\dagger\mathrm{D}_u\phi+\phi\left(\mathrm{D}_u\phi\right)^\dagger=\phi\left(\partial_u\phi\right)^\dagger+\phi^\dagger\partial_u\phi=\partial_u|\phi|^2.$$

Thus, the symmetrization in (42) yields the estimate

$$\begin{aligned}&\left|\int_{U'}^{u_*}\int_{V'}^{v_*}\mathrm{D}_u\phi\left(\partial_v\phi\right)^\dagger+\partial_v\phi\left(\mathrm{D}_u\phi\right)^\dagger\ \mathrm{d}v\mathrm{d}u\right|\leq\\ &\qquad 2\sup_{V'\leq v\leq v_*}\left|\int_{U'}^{u_*}\partial_u|\phi|^2\ \mathrm{d}u\right|+\left|\int_{U'}^{u_*}\int_{V'}^{v_*}\phi^\dagger\partial_v\mathrm{D}_u\phi+\phi\partial_v\left(\mathrm{D}_u\phi\right)^\dagger\ \mathrm{d}v\mathrm{d}u\right|.\end{aligned} \tag{43}$$

Using (41), we can immediately bound the first term on the right-hand side of (43). Thus, it remains to consider the second term.

Let us note that

$$\begin{aligned}\phi^\dagger \partial_v \mathrm{D}_u \phi &= \phi^\dagger \partial_v \partial_u \phi + \mathfrak{e}\mathrm{i}\phi^\dagger \partial_v (A_u \phi) \\ \phi \partial_v \left(\mathrm{D}_u \phi\right)^\dagger &= \phi \partial_v \left(\partial_u \phi\right)^\dagger - \mathfrak{e}\mathrm{i}\phi \partial_v (A_u \phi^\dagger).\end{aligned}$$

Using the wave equation (24), we obtain

$$\begin{aligned}\phi^\dagger \partial_u \partial_v \phi = -r^{-1}&\left(\nu \phi^\dagger \partial_v \phi + \lambda \phi^\dagger \partial_u \phi\right) \\ &-\mathfrak{e}\mathrm{i}\left(A_u |\phi|^2 \partial_v \log r + A_u \phi^\dagger \partial_v \phi - \frac{1}{2} F_{uv} |\phi|^2\right) - \frac{1}{4}\mathfrak{m}^2 \Omega |\phi|^2\end{aligned}$$

and similarly, taking the complex conjugate of (24), we have

$$\begin{aligned}\phi \partial_u \left(\partial_v \phi\right)^\dagger = -r^{-1}&\left(\nu \phi \left(\partial_v \phi\right)^\dagger + \lambda \phi \left(\partial_u \phi\right)^\dagger\right) \\ &+\mathfrak{e}\mathrm{i}\left(A_u |\phi|^2 \partial_v \log r + A_u \phi \left(\partial_v \phi\right)^\dagger - \frac{1}{2} F_{uv} |\phi|^2\right) - \frac{1}{4}\mathfrak{m}^2 \Omega^2 |\phi|^2.\end{aligned}$$

In particular,

$$\begin{aligned}&\phi^\dagger \partial_u \partial_v \phi + \phi \partial_u \left(\partial_v \phi\right)^\dagger = \\ &\quad -r^{-1}\left(\nu \partial_v |\phi|^2 + \lambda \partial_u |\phi|^2\right) + \mathfrak{e}\mathrm{i}A_u \left(\phi \left(\partial_v \phi\right)^\dagger - \phi^\dagger \partial_v \phi\right) - \frac{1}{2}\mathfrak{m}^2 \Omega^2 |\phi|^2. \qquad (44)\end{aligned}$$

Lastly, we note that

$$i\mathfrak{e}\left(\phi^\dagger \partial_v (A_u \phi) - \phi \partial_v (A_u \phi^\dagger)\right) = \mathfrak{e}\mathrm{i}A_u \left(\phi^\dagger \partial_v \phi - \phi \left(\partial_v \phi\right)^\dagger\right). \qquad (45)$$

It follows from (44) and (45) that

$$\phi^\dagger \partial_v \mathrm{D}_u \phi + \phi \partial_v \left(\mathrm{D}_u \phi\right)^\dagger = -r^{-1}\left(\nu \partial_v |\phi|^2 + \lambda \partial_u |\phi|^2\right) - \frac{1}{2}\mathfrak{m}^2 \Omega^2 |\phi|^2.$$

Using bounds (29), (33), (34) and (41), we immediately retrieve the bound we seek:

$$\left| \int_{U'}^{u_*} \int_{V'}^{v_*} \phi^\dagger \partial_v \mathrm{D}_u \phi + \phi \partial_v \left(\mathrm{D}_u \phi\right)^\dagger \ \mathrm{d}v \mathrm{d}u \right| \le C,$$

whence (43) gives

$$\left| \int_{U'}^{u_*} \int_{V'}^{v_*} \mathrm{D}_u \phi \left(\partial_v \phi\right)^\dagger + \partial_v \phi \left(\mathrm{D}_u \phi\right)^\dagger \ \mathrm{d}v \mathrm{d}u \right| \le C.$$

In particular, integrating (19) we obtain

$$\sup_{\mathcal{D}} |\log \Omega^2| \le C < \infty.$$

Thus, there are constants c_0 and c_1 such that

$$0 < c_0 \le \Omega^2 \le c_1 < \infty \qquad (46)$$

uniformly in $\mathcal{D}$.

4.3.4 One-sided estimates for $r\lambda$ and $r\nu$

Note that (18) gives

$$\partial_u(r\lambda) = \partial_v(r\nu) \geq -\frac{1}{4}\Omega^2.$$

Using the uniform estimate (46) for Ω^2, integration then yields

$$\begin{aligned} -r\lambda(u_*, v_*) &\leq -r\lambda(U', v_*) + \frac{1}{4}\int_{U'}^{u_*} \Omega^2 \, \mathrm{d}u \\ &\leq \Lambda + \frac{1}{4}c_1(U - U') =: \Lambda' < \infty. \end{aligned} \tag{47}$$

Similarly, we note that

$$-r\nu(u_*, v_*) \leq N + \frac{1}{4}c_1(V - V') =: N' < \infty.$$

4.3.5 L^2-estimates for $\partial_v\phi$ and $\mathrm{D}_u\phi$

Integrating (21) we obtain

$$\begin{aligned} \Omega^{-2}\lambda(u_*, V') - \Omega^{-2}\lambda(u_*, v_*) &= \int_{V'}^{v_*} 4\pi r\Omega^{-2}|\partial_v\phi|^2 \, \mathrm{d}v \\ &\geq 4\pi r_0 c_1 \int_{V'}^{v_*} |\partial_v\phi|^2 \, \mathrm{d}v. \end{aligned}$$

Thus, we have, using the one-sided estimate (47),

$$\begin{aligned} \int_{V'}^{v_*} |\partial_v\phi|^2 \, \mathrm{d}v &\leq \frac{1}{4\pi r_0 c_1}\left(\Omega^{-2}\lambda(u_*, V') - \Omega^{-2}\lambda(u_*, v_*)\right) \\ &\leq \frac{c_0}{4\pi r_0^2 c_1}\left(\Lambda + \Lambda'\right). \end{aligned} \tag{48}$$

Similarly, we have

$$\int_{U'}^{u_*} |\mathrm{D}_u\phi|^2 \, \mathrm{d}u \leq \frac{c_0}{4\pi r_0^2 c_1}\left(N + N'\right). \tag{49}$$

4.3.6 Uniform bound on T_{uv}

With bounds on ϕ and Ω^2, it remains to bound Q_e in order to give, from (17), an estimate for T_{uv}.

Using Cauchy-Schwarz, (41) and (48) we note the bound

$$\left|\int_{V'}^{v_*} r\left(\phi\left(\partial_v\phi\right)^\dagger - \phi^\dagger\partial_v\phi\right) \, \mathrm{d}v\right| \leq 2\left(\int_{V'}^{v_*} |\partial_v\phi|^2 \, \mathrm{d}v\right)^{\frac{1}{2}}\left(\int_{V'}^{v_*} |r\phi|^2 \, \mathrm{d}v\right)^{\frac{1}{2}} \leq C.$$

Upon integration of (23), we therefore obtain

$$|Q_\mathrm{e}(u_*, v_*)| \leq |Q_\mathrm{e}(u_*, V')| + \left|\int_{V'}^{v_*} \mathfrak{e}\mathrm{i}\sqrt{\pi}r^2\left(\phi\left(\partial_v\phi\right)^\dagger - \phi^\dagger\partial_v\phi\right) \, \mathrm{d}v\right| \leq B + |\mathfrak{e}|\sqrt{\pi}RC. \tag{50}$$

It thus follows that

$$\sup_{\mathcal{D}} T_{uv} \leq C.$$

4.3.7 Remaining uniform estimates

We can now bound all other quantities quite easily.

To estimate the first derivatives of ϕ let us use the wave equation (24) to compute

$$\partial_u\theta_A + \mathfrak{e}\mathrm{i}A_u\theta_A = -\frac{\zeta_A}{r}\lambda - \frac{1}{4}\Omega^2\left(\mathfrak{m}^2 r\phi - \mathfrak{e}\mathrm{i}2\sqrt{\pi}r^{-1}Q_{\mathrm{e}}\phi\right) \tag{51}$$

$$\partial_v\zeta_A = -\frac{\theta_A}{r}\nu - \frac{1}{4}\Omega^2\left(\mathfrak{m}^2 r\phi + \mathfrak{e}\mathrm{i}2\sqrt{\pi}r^{-1}Q_{\mathrm{e}}\phi\right). \tag{52}$$

We note that (47) and (49) gives

$$\left|\int_{U'}^{u_*} -\frac{\zeta_A}{r}\lambda\ \mathrm{d}u\right| \le C\int_{U'}^{u_*}|\mathrm{D}_u\phi|\ \mathrm{d}u \le C\left(\int_{U'}^{u_*}|\mathrm{D}_u\phi|^2\ \mathrm{d}u\right)^{\frac{1}{2}}\left(\int_{U'}^{u_*}1\ \mathrm{d}u\right)^{\frac{1}{2}} \le C.$$

Integrating (51), we therefore have

$$\sup_{\mathcal{D}}|\theta_A| \le C$$

using the estimates on ϕ, Ω^2 and Q_{e}. Using this bound, we integrate (52) and similarly obtain

$$\sup_{\mathcal{D}}|\zeta_A| \le C.$$

From the uniform bounds on T_{uv} and Ω^2 we note that (32) gives

$$\sup_{\mathcal{D}}|r\lambda| \le C.$$

Similarly, we have

$$\sup_{\mathcal{D}}|r\nu| \le C.$$

Integration of (27) also gives, using (50), the uniform estimate

$$\sup_{\mathcal{D}}|A_u| \le C.$$

It remains now to show that $\partial_u\Omega$, $\partial_v\Omega$, $\partial_u\nu$ and $\partial_v\lambda$ are bounded quantities in $\mathcal{D}$. By integrating (19), we obtain uniform estimates on $\partial_u\Omega$ and $\partial_v\Omega$ using already derived bounds. Then, we can give uniform estimates for $\partial_u\nu$ and $\partial_v\lambda$ by using (20) and (21), respectively.

We have established that $N(\mathcal{D}) < \infty$ and Theorem 1.11 follows from applying Proposition 4.2. □

5 Proof of Theorem 1.1: global characterization of spacetime

With the extension principle of Theorem 1.11 having been proven, we are now able to give a characterization of first singularities that arise in the collapse of self-gravitating charged scalar fields. Together with monotonicity arguments, a consequence of the dominant energy condition (although, in some cases, monotonicity arising from the weaker null energy condition will suffice), we can then give a proof of Theorem 1.1, establishing the global characterization of spacetime.

5.1 Characterization of first singularities

We apply Theorem 1.11 under the assumptions of initial data in Theorem 1.1 to give a characterization of first singularities $\mathcal{B}_1^+$. Introducing the notation

$$r_{\inf}(p) = \lim_{q\to p} \inf_{J^-(p)\cap J^+(q)\cap\mathcal{Q}^+} r(q)$$

for $p \in \overline{\mathcal{Q}^+}$ and $q \in J^-(p)\cap\mathcal{Q}^+$, we have, as an immediate corollary of Theorem 1.11, recalling Proposition 3.2,

Corollary 5.1. *If $p \in \mathcal{B}_1^+$, then $r_{\inf}(p) = 0$.*

As a consequence of Corollary 5.1, if $p \in \overline{\mathcal{R}}\backslash\overline{\Gamma} \subset \overline{\mathcal{Q}^+}$ and $q \in \left(I^-(p)\cap\overline{\mathcal{R}}\right)\backslash\{p\}$ are such that

$$\mathcal{D} = \left(J^-(p)\cap J^+(q)\right)\backslash\{p\} \subset \mathcal{R}\cup\mathcal{A},$$

then $p \notin \mathcal{B}_1^+$, for $r \geq r_0 > 0$ on $\mathcal{D}$ since $\lambda \geq 0$ and $\nu < 0$ in $\mathcal{R}\cup\mathcal{A}$. Whence, we state

Corollary 5.2. *If $p \in \mathcal{B}_1^+\cap\overline{\mathcal{R}}$, then $p = b_\Gamma$.*

In particular, for the Einstein-Maxwell-Klein-Gordon system the generalized extension principle implies the weak extension principle (cf. §1.8.1). More generally, since Proposition 3.2 holds for any matter model that obeys the null energy condition, this establishes Proposition 1.1 of §1.8.2, i.e. every strongly tame Einstein-matter system is also weakly tame.

5.2 The Hawking mass and its monotonicity properties

It will be convenient here to recall that the Hawking mass m is given by

$$1 - \frac{2m}{r} = g^{ab}\partial_a r\partial_b r = -4\Omega^{-2}\lambda\nu, \tag{53}$$

and its evolution equations can be shown to satisfy (cf. [18])

$$\begin{aligned}\partial_u m &= 8\pi r^2\Omega^{-2}\left(T_{uv}\nu - T_{uu}\lambda\right)\\ \partial_v m &= 8\pi r^2\Omega^{-2}\left(T_{uv}\lambda - T_{vv}\nu\right).\end{aligned}$$

Because we assume that $\mathfrak{m}^2 \geq 0$, the Einstein-Maxwell-Klein-Gordon system satisfies the dominant energy condition (cf. §2.2.2):

$$T_{uv} \geq 0, \quad T_{uu} \geq 0 \quad \text{and} \quad T_{vv} \geq 0.$$

As a consequence of Proposition 3.1, the Hawking mass therefore satisfies monotonicity properties

$$\partial_v m \geq 0 \quad \text{and} \quad \partial_u m \leq 0 \tag{54}$$

in $\mathcal{R}\cup\mathcal{A}$. This monotonicity is only used in a handful of the proofs below. Where it is not exploited (keeping in mind the need for the dominant energy condition[51] in Corollary 5.1), the null energy condition

$$T_{uu} \geq 0 \quad \text{and} \quad T_{vv} \geq 0$$

will suffice.

5.3 Statements I and II

Recall the notation of §3. We will systematically refine the boundary characterization of Proposition 3.4 so as to obtain finally that which is given in Theorem 1.1.

[51]see, however, footnote 50

5.3.1 Null segment $\mathcal{N}_\Gamma$ emanating from b_Γ

Let

$$\mathcal{CH}_\Gamma = \{q \in \mathcal{N}_\Gamma : \exists p, p' \in \mathcal{N}_\Gamma \ \text{ s.t. } \ q \in (p, p'] \ \text{ and } \ r_{\inf}(q') \neq 0 \ \forall q' \in (p, p')\},$$

and define

$$\mathcal{S}_\Gamma = \mathcal{N}_\Gamma \backslash \mathcal{CH}_\Gamma.$$

We note that $r_{\inf}$ need not, *a priori*, vanish on $\mathcal{S}_\Gamma$. By definition, however, for every $(U, v) \in \mathcal{S}_\Gamma$ and every $\delta > 0$, there exists $v - \delta < \tilde{v} \leq v$ such that $(U, \tilde{v}) \in \mathcal{S}_\Gamma$ and $r_{\inf}(U, \tilde{v}) = 0$.

We show $\mathcal{CH}_\Gamma$ is a connected set. Suppose, on the contrary, that $\mathcal{CH}_\Gamma$ is disconnected. Then, there exists $(U, v) \in \mathcal{CH}_\Gamma$ and $(U, v') \in \mathcal{CH}_\Gamma$, where without loss of generality $v < v'$, and $v < v'' < v'$ such that $(U, v'') \in \mathcal{S}_\Gamma$. Moreover, there exists $v < v''' \leq v''$ such that $r_{\inf}(U, v''') = 0$. Thus, there exists a sequence $(U_j, v_j''') \to (U, v''')$ such that $r(U_j, v_j''') \to 0$ with $U_j \leq U$ and $v_j''' \leq v'''$. Define

$$r_0 = \min\{r_{\inf}(U, v), r_{\inf}(U, v')\} > 0.$$

Choose J sufficiently large so that $r(U_j, v_j''') < r_0$ for all $j \geq J$. Since $\nu < 0$ in $\mathcal{Q}^+$, it follows that

$$r(u, v) > r_0 \qquad \text{and} \qquad r(u, v') > r_0$$

for all $u \geq U_J$. It follows that for each $U_j \in [U_J, U)$ there exists $\tilde{v}(U_j) \in (v, v_j''')$ and $\hat{v}(U_j) \in (v_j''', v')$ such that

$$\lambda(U_j, \tilde{v}(U_j)) < 0 \qquad \text{and} \qquad \lambda(U_j, \hat{v}(U_j)) > 0.$$

This, however, contradicts monotonicity (21) and no such $\hat{v}(U_j)$ can exist.

We conclude that $\mathcal{CH}_\Gamma$ is a single (possibly empty) half-open interval. From monotonicity (21), it then follows that $\mathcal{N}_\Gamma$ is given by

$$\mathcal{N}_\Gamma = \mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2,$$

where $\mathcal{S}_\Gamma^1$ is a half-open (possibly empty) connected component of $\mathcal{S}_\Gamma$ that emanates from (but does not include) b_Γ and $\mathcal{S}_\Gamma^2$ is a half-open (possibly empty) connected component of $\mathcal{S}_\Gamma$ that emanates from (but does not include) the future endpoint of $\mathcal{CH}_\Gamma$. We note that if $\mathcal{CH}_\Gamma = \emptyset$, then, necessarily, $\mathcal{S}_\Gamma^2 = \emptyset$.

5.3.2 Null segment $\mathcal{N}_{i^+}$ emanating from $i^\square$

Let

$$\mathcal{CH}_{i^+} = \{q \in \mathcal{N}_{i^+} : \exists p, p' \in \mathcal{N}_{i^+} \ \text{ s.t. } \ q \in (p, p'] \ \text{ and } \ r_{\inf}(q') \neq 0 \ \forall q' \in (p, p')\}$$

and define

$$\mathcal{S}_{i^+} = \mathcal{N}_{i^+} \backslash \mathcal{CH}_{i^+}.$$

Again we note that $r_{\inf}$ need not, *a priori*, vanish on $\mathcal{S}_{i^+}$. By definition, however, for every $(u, V) \in \mathcal{S}_{i^+}$ and every $\delta > 0$, there exists $u - \delta < \tilde{u} \leq u$ such that $(\tilde{u}, V) \in \mathcal{S}_{i^+}$ and $r_{\inf}(\tilde{u}, V) = 0$.

We will establish that $\mathcal{S}_{i^+}$ is a connected set by showing that if $(u, V) \in \mathcal{S}_{i^+}$ and $r_{\inf}(u, V) = 0$, then $r_{\inf}(u', V) = 0$ for all $u' \geq u$. It will then follow that $\mathcal{CH}_{i^+}$ is a half-open (possibly empty) interval that, necessarily, emanates from (but does not include) $i^\square$. $\mathcal{S}_{i^+}$ is then a half-open (possibly empty) interval that emanates from (but does not include) the future endpoint of $\mathcal{CH}_{i^+} \cup i^\square$.

Suppose $(u, V) \in \mathcal{S}_{i^+}$. Then, by definition, there exists some $u' \leq u$ such that $(u', V) \in \mathcal{S}_{i^+}$ and a sequence $(u_j', V_j) \to (u', V)$ such that $r(u_j', V_j) \to 0$ with $u_j \leq u'$ and $V_j < V$. Since $\nu < 0$ in $\mathcal{Q}^+$, $r(u'', V_j) \leq r(u_j', V_j)$ for all $\{(u'', V_j) : u'' \geq u'\} \cap \mathcal{Q}^+$. Thus, $(u'', V) \in \mathcal{S}_{i^+}$ and, in particular, *a posteriori*, $r_{\inf}$ vanishes on $\mathcal{S}_{i^+}$.

5.3.3 The remaining achronal boundary

Let us define

$$\mathcal{S} = \bigcup_{p\in\mathcal{B}_1^+\setminus b_\Gamma} \{p\} \cup \mathcal{N}_p^1 \cup \mathcal{N}_p^2.$$

We will show that $r_{\mathrm{inf}} = 0$ on $\mathcal{S}$.

Recall that Corollary 5.1 gives that $r_{\mathrm{inf}}(u,v) = 0$ for all $(u,v) \in \mathcal{B}_1^+\setminus b_\Gamma$. In particular, there exists a sequence $(u_j, v_j) \to (u,v)$ such that $r(u_j,v_j) \to 0$ with $u_j \le u$ and $v_j \le v$. Let $\mathcal{N}_p^1 \subset \{v = v(p)\}$. Since $\nu < 0$ in $\mathcal{Q}^+$, $r(u', v_j) \le r(u_j, v_j)$ for all $u' \ge u$. Thus, $r_{\mathrm{inf}}(u', v) = 0$. Let $\mathcal{N}_p^2 \subset \{u = u(p)\}$. Since (u,v) is *non-central*, there exists, by compactness, a $v' < v$ such that $(u, v') \in \mathcal{Q}^+$ and an $\epsilon > 0$ such that $r(u', v') > \epsilon$ for all $\{(u', v') : u' \le u\}\cap\mathcal{Q}^+$. Moreover, there exists a sufficiently large J such that $r(u_j, v_j) < \epsilon$ for all $j \ge J$. It follows that there exists a $\hat{v}(j) \in (v', v_j)$ such that $\lambda(u_j, \hat{v}(j)) < 0$. Appealing to monotonicity (21), we conclude that $\lambda(u_j, v'') < 0$ for all $v'' \ge \hat{v}(j)$. Whence $r(u_j, v'') \le r(u_j, v_j)$ and $r_{\mathrm{inf}}(u, v'') = 0$.

5.3.4 $\mathcal{I}^+$ is an open set in the topology of $\mathcal{B}^+$

If $b_\Gamma = i^+$, then the openness of $\mathcal{I}^+$ is obvious by definition. If $b_\Gamma \ne i^\square$, it follows from Statement VI, which is proven below in §5.7, that $i^\square \notin \mathcal{I}^+$. This will establish the claim.

5.3.5 Continuous extendibility of r

Of the remaining properties of the boundary to establish is that of the extendibility properties of r. We do this in the sequel.

r extends continuously to zero on $(\mathcal{S} \cup \mathcal{S}_{i^+})\setminus b_\Gamma$

For every $(u,v) \in (\mathcal{S} \cup \mathcal{S}_{i^+})\setminus b_\Gamma$, there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ such that $\mathcal{U}\cap\mathcal{Q}^+ \subset \mathcal{T}$. Moreover, we note that $r_{\mathrm{inf}}(u,v) = 0$. Thus, there exists a sequence $(u_j, v_j) \to (u,v)$ such that $r(u_j, v_j) \to 0$ with $u_j \le u$ and $v_j \le v$. Perturbing the sequence, we may, without loss of generality, assume that $v_j < v$. Given $\epsilon > 0$, there exists a sufficiently large J such that $r(u_j, v_j) < \epsilon$ for all $j \ge J$. Since $\nu < 0$ and $\lambda < 0$ in $\mathcal{U}\cap\mathcal{Q}^+$, it follows that $r(u', v') < \epsilon$ in $\{u' \ge u_J\} \cap \{v' \ge v_J\} \cap \mathcal{U} \cap \mathcal{Q}^+$.

r extends continuously to zero on $b_\Gamma \in \mathcal{S}_{i^+}$

Let $(U, V) = b_\Gamma \in \mathcal{S}_{i^+}$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of b_Γ.

Fix $u < U$ such that $(u', V) \in \mathcal{U} \cap \mathcal{S}_{i^+}$. Since r extends continuously to zero on (u', V), there exists a sequence (u'_j, V_j) such that $r(u'_j, V_j) \to 0$ with $u'_j \le u'$ and $V_j < V$. Given $\epsilon > 0$, there exists a sufficiently large J such that $r(u'_j, V_j) < \epsilon$ for all $j \ge J$. Since there exists a sufficiently small neighborhood $\widetilde{\mathcal{U}}$ of (u', V) such that $\widetilde{\mathcal{U}}\cap\mathcal{Q}^+ \subset \mathcal{T}$, we can assume, without loss of generality, that $(u'_J, V_J) \in \mathcal{T}$. By monotonicity (21), it follows that $(u'_J, v') \in \mathcal{T}$ for all $\{v' \ge V_J\} \cap \mathcal{Q}^+$. Since $\nu < 0$ in $\mathcal{Q}^+$, we have that

$$r(u'', v) \le \epsilon$$

for all $(u'', v) \in \{u'' \ge u'_J\} \cap \{v \ge V_J\} \cap \mathcal{U} \cap \mathcal{Q}^+$.

r extends continuously to zero on $b_\Gamma\setminus\left(\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square\right)$

Let $(U, V) = b_\Gamma\setminus\left(\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square\right)$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of b_Γ. Since, by assumption, b_Γ does not coincide with the future endpoint of $\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square$, we note that there exists a sufficiently small $\delta > 0$ such that $\{|v - V| \le \delta\} \cap \mathcal{Q}^+$ is compact. Hence, for all

$u_0 \le u < U$, there exists a constant $C(u_0) > 0$ such that, by (53) and monotonicity (20), we have

$$-\frac{1}{4}\left(\Omega^{-2}\nu\right)^{-1}(u,v) = \frac{\lambda}{1-\frac{2m}{r}}(u,v) \le \frac{\lambda}{1-\frac{2m}{r}}(u_0,v) \le C(u_0) \tag{55}$$

in $\{|v-V| \le \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$. Since (53) gives $1-\frac{2m}{r} \le 1$ in $\mathcal{R}\cup\mathcal{A}$ (cf. Statement V), we therefore obtain

$$0 \le \lambda(u,v) \le \frac{\lambda}{1-\frac{2m}{r}}(u_0,v)\left(1-\frac{2m}{r}\right)(u,v) \le C, \tag{56}$$

for all $(u,v) \in \{|v-V| \le \delta\} \cap \mathcal{U} \cap (\mathcal{R}\cup\mathcal{A})$.

Given $\epsilon > 0$, let us choose $\delta' \le \min\{\delta, (2C)^{-1}\epsilon\}$. Along each outgoing null segment in $\mathcal{U}\cap\mathcal{Q}^+$ emanating from $\Gamma = \{(u, v_\Gamma(u))\}$ let us define

$$v_{\mathcal{A}}(u) = \begin{cases} \inf\{v : \lambda(u,v) = 0\}, & \text{if } \mathcal{A}\cap\{v \le V+\delta'\} \neq \emptyset; \\ V+\delta', & \text{if } \mathcal{A}\cap\{v \le V+\delta'\} = \emptyset. \end{cases}$$

We note that $v_\Gamma(u) < v_{\mathcal{A}}(u)$ since $\Gamma \subset \mathcal{R}$, which is proven below in §5.5, and thus $\{u\}\times[v_\Gamma(u), v_{\mathcal{A}}(u)) \subset \mathcal{R}$. Let $(u', v_\Gamma(u'))$ be the intersection point of $\{v = V-\delta'\}$ with Γ. Then,

$$|r(u,v) - r(u,v_\Gamma(u))| = \int_{v_\Gamma(u)}^{v} \lambda(u,\overline{v})\, \mathrm{d}\overline{v} \le C(v - v_\Gamma(u)) \le 2C\delta' \le \epsilon$$

in $\{v \le v_{\mathcal{A}}(u)\}\cap\{u \ge u'\}\cap\mathcal{U}\cap(\mathcal{R}\cup\mathcal{A})$. Since r is decreasing in $\mathcal{T}$, there exists a neighborhood $\mathcal{U}' \subset \mathbb{R}^{1+1}$ satisfying $\mathcal{U}' \subset \{|v-V| \le \delta'\}\cap\{u \ge u'\}\cap\mathcal{U}$ so that

$$|r(u,v) - r(U,V)| \le \epsilon$$

for all $(u,v) \in \mathcal{U}'\cap\mathcal{Q}^+$.

r extends continuously to zero on $\mathcal{S}^1_\Gamma \setminus \left(\mathcal{CH}_{i^+} \cup i^\square\right)$

Let $(U,v) \in \mathcal{S}^1_\Gamma \setminus \left(\mathcal{CH}_{i^+} \cup i^\square\right)$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$. If $(U,v) \in \mathcal{S}_{i^+}$, then we establish continuity as in the case in which $b_\Gamma \in \mathcal{S}_{i^+}$. Thus, without loss of generality, let us assume that $(U,v) \notin \mathcal{S}_{i^+}$. Since $\mathcal{S}^1_\Gamma$ does not coincide with the future endpoint of $\mathcal{S}_{i^+}\cup\mathcal{CH}_{i^+}\cup i^\square$, we note that there exists a sufficiently small $\delta > 0$ such that $\{|v'-v| \le \delta\}\cap\mathcal{Q}^+$ is compact. Hence, for all $u_0 \le u < U$, there exists a constant $C(u_0) > 0$ such that, by (53) and monotonicity (20), we have

$$-\frac{1}{4}\left(\Omega^{-2}\nu\right)^{-1}(u,v') = \frac{\lambda}{1-\frac{2m}{r}}(u,v') \le \frac{\lambda}{1-\frac{2m}{r}}(u_0,v') \le C(u_0) \tag{57}$$

in $\{|v'-v| \le \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$. Since (53) gives $1-\frac{2m}{r} \le 1$ in $\mathcal{R}\cup\mathcal{A}$ (cf. Statement V), we therefore obtain

$$0 \le \lambda(u,v') \le \frac{\lambda}{1-\frac{2m}{r}}(u_0,v')\left(1-\frac{2m}{r}\right)(u,v') \le C, \tag{58}$$

for all $(u,v') \in \{|v'-v| \le \delta\} \cap \mathcal{U} \cap (\mathcal{R}\cup\mathcal{A})$.

Let $\epsilon > 0$ be given and choose $0 < \delta' \le \delta$ such that

$$C\delta' < \frac{1}{4}\epsilon.$$

By definition, there exists some $v - \delta' < v'' \le v$ such that $(U,v'') \in \mathcal{S}^1_\Gamma$ and a sequence $(U_j, v''_j) \to (U, v'')$ such that $r(U_j, v''_j) \to 0$ with $U_j \le U$ and $v''_j \le v''$. In particular, there exists a sufficiently large J such that $r(U_j, v''_j) < \frac{1}{2}\epsilon$ for all $j \ge J$. We then have

$$r(U_j, v''') = r(U_j, v''_j) + \int_{v''_j}^{v'''} \lambda(U_j, \overline{v})\, \mathrm{d}\overline{v} \le \frac{1}{2}\epsilon + 2C\delta' \le \epsilon$$

for all $(U_j, v''') \in \{v''_j \le v''' \le v + \delta'\} \cap \mathcal{U} \cap (\mathcal{R} \cup \mathcal{A})$. Since $\nu < 0$ in $\mathcal{Q}^+$, it then follows that

$$r(u, v''') < \epsilon$$

in $\{v'' \le v''' \le v + \delta'\} \cap \{u \ge U_J\} \cap \mathcal{U} \cap (\mathcal{R} \cup \mathcal{A})$. Since r is decreasing in $\mathcal{T}$, there is a neighborhood $\mathcal{U}' \subset \mathbb{R}^{1+1}$ satisfying $\mathcal{U}' \subset \{v'' \le v' \le v + \delta'\} \cap \{u \ge U_J\} \cap \mathcal{U}$ such that

$$|r(u, v') - r(U, v)| \le \epsilon$$

for all $(u, v') \in \mathcal{U}' \cap \mathcal{Q}^+$.

r extends continuously to zero on $\mathcal{S}^2_\Gamma$

For every $(U, v) \in \mathcal{S}^2_\Gamma$, there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ such that $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{T}$. By definition, there exists a $v' \le v$ such that $(U, v') \in \mathcal{S}^2_\Gamma \cap \mathcal{U}$ and a sequence $(U_j, v'_j) \to (U, v')$ such that $r(U_j, v'_j) \to 0$ with $U_j \le U$ and $v'_j \le v'$. Given $\epsilon > 0$, there exists a sufficiently large J such that $r(U_j, v'_j) < \epsilon$ for all $j \ge J$. Since $\lambda < 0$ and $\nu < 0$ in $\mathcal{U} \cap \mathcal{Q}^+$, we therefore have

$$r(u, v'') < \epsilon$$

in $\mathcal{U}' \cap \mathcal{Q}^+$ for any neighborhood $\mathcal{U}'$ satisfying $\mathcal{U}' \subset \{v'' \ge v'\} \cap \{u \ge U_J\} \cap \mathcal{U}$.

r extends continuously on int$(\mathcal{CH}_\Gamma)$

Let $(U, v) \in \mathrm{int}(\mathcal{CH}_\Gamma)$. Suppose $\mathcal{U} \subset \mathbb{R}^{1+1}$ is a neighborhood of (U, v).

Re-writing (18) in terms of the Hawking mass and (17), we note that in the trapped region

$$\partial_u(-\lambda) = \frac{2}{r^2}\left(m - \frac{2\pi\sigma^2}{r}\right)\frac{\lambda}{1 - \frac{2m}{r}}(-\nu) \le \frac{2}{r^2}\frac{m\lambda}{1 - \frac{2m}{r}}(-\nu), \tag{59}$$

where

$$\sigma^2 = 4\Omega^{-2}r^4 T_{uv} \ge 0.$$

Note that by (53) we have $\frac{2m}{r} > 1$ in $\mathcal{T}$ (cf. Statement V). Define the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ by

$$\mathcal{U}_1 = \left\{\frac{2m}{r} \ge 2\right\} \cap \mathcal{U} \cap \mathcal{T} \qquad \text{and} \qquad \mathcal{U}_2 = \left\{\frac{2m}{r} \le 2\right\} \cap \mathcal{U} \cap \mathcal{T}.$$

Integrating the right-hand side of (59) along segments $([u_0, u] \times \{v'\}) \cap \mathcal{U} \cap \mathcal{T}$, we obtain

$$\int_{u_0}^{u} \frac{2}{r^2}\frac{m\lambda}{1 - \frac{2m}{r}}(-\nu)\, \mathrm{d}\overline{u} = \int_{[u_0,u]\cap\mathcal{U}_1} \frac{|\lambda|(-\nu)}{r}\frac{\frac{2m}{r}}{\frac{2m}{r} - 1}\, \mathrm{d}\overline{u} + \int_{[u_0,u]\cap\mathcal{U}_2} \frac{2m}{r}\frac{\lambda}{1 - \frac{2m}{r}}\frac{-\nu}{r}\, \mathrm{d}\overline{u}.$$

On the set $\mathcal{U}_1$, we note that

$$0 < \frac{\frac{2m}{r}}{\frac{2m}{r} - 1} \le 2.$$

Thus,

$$\int_{[u_0,u]\cap\mathcal{U}_1} \frac{|\lambda|(-\nu)}{r}\frac{\frac{2m}{r}}{\frac{2m}{r} - 1}\, \mathrm{d}\overline{u} \le 2\int_{[u_0,u]\cap\mathcal{U}_1} \frac{|\lambda|(-\nu)}{r}\, \mathrm{d}\overline{u}.$$

In view of the assumption that (U, v) is in the interior of $\mathcal{CH}_\Gamma$, given $C(u_0) > 0$, there exists a sufficiently small $\delta > 0$, recalling (57), such that

$$0 < \frac{\lambda}{1 - \frac{2m}{r}}(u, v') \le \frac{\lambda}{1 - \frac{2m}{r}}(u_0, v') \le C(u_0)$$

in the compact set $\{|v'-v|\leq\delta\}\cap\mathcal{U}\cap\mathcal{Q}^+$. Moreover, by compactness and since $\nu<0$ in $\mathcal{Q}^+$, it follows that

$$r > r_0 = \inf_{\{|v'-v|\leq\delta\}} r_{\inf}(U,v') > 0$$

in $\{|v'-v|\leq\delta\}\cap\mathcal{U}\cap\mathcal{Q}^+$. Thus, we have

$$\int_{[u_0,u]\cap\mathcal{U}_2} \frac{2m}{r}\frac{\lambda}{1-\frac{2m}{r}}\frac{-\nu}{r}\,\mathrm{d}\overline{u} \leq C(u_0)\int_{[u_0,u]\cap\mathcal{U}_2}\frac{-\nu}{r}\,\mathrm{d}\overline{u} \leq C(u_0,r_0)<\infty.$$

It follows, from integration of (59), that

$$|\lambda(u,v')| \leq C(u_0,r_0) + 2\int_{[u_0,u]\cap\mathcal{U}_1}\frac{|\lambda|(-\nu)}{r}\,\mathrm{d}\overline{u}.$$

Applying a Grönwall inequality then yields

$$|\lambda(u,v')| \leq C(u_0,r_0)\exp\left(\int_{[u_0,u]\cap\mathcal{U}_1}\frac{-\nu}{r}\,\mathrm{d}\overline{u}\right),$$

whence we obtain

$$\sup_{\{|v'-v|\leq\delta\}\cap\mathcal{U}\cap\mathcal{T}} |\lambda(u,v')| \leq C(u_0,r_0)<\infty.$$

Together with (58), this gives a uniform bound $|\lambda(u,v')|\leq C$ in $\{|v'-v|\leq\delta\}\cap\mathcal{U}\cap\mathcal{Q}^+$. Continuity of r is now easily established.

r extends continuously to ∞ on $\mathcal{I}^+$

To establish continuity at $(u,V)\in\mathcal{I}^+$, it suffices to show that there exists a neighborhood $\mathcal{U}\subset\mathbb{R}^{1+1}$ such that ν is uniformly bounded in $\mathcal{U}\cap\mathcal{Q}^+$.

Note that (59) gives

$$\partial_v\log(-\nu) = \frac{2}{r^2}\frac{\lambda}{1-\frac{2m}{r}}\left(m-\frac{2\pi\sigma^2}{r}\right). \tag{60}$$

Since $\lambda>0$ in $J^-(\mathcal{I}^+)$, monotonicity (54) implies that m extends to a bounded function along $\mathcal{I}^+$ (cf. Statement V). Let

$$M = \sup_{\mathcal{U}\cap\mathcal{I}^+} m < \infty.$$

For r sufficiently large (noting that $r\to\infty$ as $v\to V$), i.e. for $\mathcal{U}$ sufficiently small, we can ensure that

$$\frac{m}{1-\frac{2m}{r}} \leq \frac{3}{2}M$$

in $\mathcal{U}\cap\mathcal{Q}^+$. In particular, we then have

$$\partial_v\log(-\nu) \leq \frac{2}{r^2}\frac{\lambda m}{1-\frac{2m}{r}} \leq 3M\frac{\lambda}{r^2}. \tag{61}$$

The result follows immediately upon integration of (61).

r extends continuously to ∞ on i^0

Continuity of r on i^0 follows immediately from an appropriate notion of 'asymptotically flat' in Theorem 1.1.

5.4 Statement III

Follows from the results given by Dafermos in [28].

5.5 Statement IV

5.5.1 Claim 1

Consider the vector field $\mathbf{T}$ on $\mathcal{Q}^+$ given by

$$\mathbf{T} = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right).$$

Since Γ is the timelike boundary of $\mathcal{Q}^+$ on which $r = 0$, we compute

$$0 = 2\mathbf{T}r|_{\Gamma} = (\nu + \lambda)\,|_{\Gamma}.$$

In particular, since $\nu < 0$ in $\mathcal{Q}^+$, it follows that

$$\lambda|_{\Gamma} > 0.$$

This establishes the claim.

5.5.2 Claims 2–3

The claims follow immediately from monotonicity (21). See [18].

5.5.3 Claim 4

Let $(u, V) \in \operatorname{int}(\mathcal{CH}_{i^+})$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of (u, V) such that either $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A}$ or $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{T}$. To establish continuity, it suffices to show that ν is uniformly bounded in $\mathcal{U} \cap \mathcal{Q}^+$.

If $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A}$, we simply note that r is constant along outgoing null segments in $\mathcal{U} \cap \mathcal{Q}^+$. It follows that ν is uniformly bounded in this case.

From (53) and monotonicity (21), we note that ν and $1 - \frac{2m}{r}$ have the same sign in the trapped region and, moreover, that

$$\partial_v\left(\frac{\nu}{1 - \frac{2m}{r}}\right) \leq 0. \tag{62}$$

If $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{T}$, we retrieve a uniform bound on ν by proceeding exactly as in the case in which we established the uniform bound on λ in the trapped region when proving the extendibility of r on $\operatorname{int}(\mathcal{CH}_\Gamma)$. We simply note that $\partial_u \lambda = \partial_v \nu$ and that monotonicity (62) takes the role of (57). Moreover, we note that since $\lambda < 0$ in $\mathcal{T}$, this monotonicity takes the role of $\nu < 0$ in ensuring that r is bounded from below away from zero in $\mathcal{U} \cap \mathcal{Q}^+$.

Remark 1. We do not say anything about the case in which $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A} \cup \mathcal{T}$, as there is no possible uniform control on $\frac{\nu}{1 - \frac{2m}{r}}$.

5.5.4 Claim 5

(a) If $\mathcal{A} \neq \emptyset$, then the limit points of $\mathcal{A}$ on $b_\Gamma \cup \mathcal{S}^1_\Gamma \cup \mathcal{CH}_\Gamma$ necessarily form a (possibly degenerate) connected closed interval. This follows from monotonicity given in Claim 2. On the other hand, there is, *a priori*, no characterization of the limit points of $\mathcal{A}$ on $\mathcal{CH}_{i^+} \cup i^+$ (cf. the diagram given in the statement of Theorem 1.1).

(b)–(e) The claims follow from monotonicity given in Claim 2 (cf. §5.3.1).

5.6 Statement V

5.6.1 Claim 1

The claim is given by (54).

5.6.2 Claim 2

The claim follows immediately from (53) since $\lambda \geq 0$ and $\nu < 0$ in $\mathcal{R} \cup \mathcal{A}$.

5.7 Statement VI

The statement follows from the results given by Dafermos in [28].

5.8 Statement VII

In spherically symmetric spacetimes, the Kretschmann scalar satisfies

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 4K^2 + \frac{4}{r^4}\left(\frac{2m}{r}\right)^2 + \frac{12}{r^2} g\left(\nabla\nabla r, \nabla\nabla r\right),$$

where K is the Gaussian curvature of the quotient metric g_{ab}, which is given by

$$K = 4\Omega^{-2}\left(\Omega^{-1}\partial_u\partial_v\Omega - \Omega^{-2}\partial_u\Omega\partial_v\Omega\right).$$

One can compute (see the appendix of [32]),

$$g\left(\nabla\nabla r, \nabla\nabla r\right) = 2\left(\frac{m}{r^2} + 2\pi r \mathrm{tr}T\right)^2 + 8\pi^2 r^2 g\left(T - \frac{1}{2} g \mathrm{tr}T, T - \frac{1}{2} g \mathrm{tr}T\right),$$

where $\mathrm{tr}T = g^{ab}T_{ab}$. The last term on the right-hand side is manifestly non-negative if the energy momentum tensor obeys the null energy condition. In this case, the Kretschmann scalar satisfies

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \geq \frac{4}{r^4}\left(\frac{2m}{r}\right)^2. \tag{63}$$

5.8.1 Claim 1

For every $p \in \mathcal{S}^2_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}$, there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of p such that $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{T}$. Recalling that $\frac{2m}{r} > 1$ in the trapped region, we have from (63),

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} > \frac{4}{r^4}.$$

For every sequence $\{p_j\}_{j=1}^{\infty} \subset \mathcal{U} \cap \mathcal{Q}^+$ with $p_j \to p$, we therefore obtain

$$\lim_{j\to\infty} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}(p_j) > \lim_{j\to\infty} \frac{4}{r^4}(p_j) = \infty.$$

This establishes the claim.

5.8.2 Claim 2

(a) Let $(U,v) \in \mathcal{S}^1_\Gamma$ and consider a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of (U,v). If (U,v) is a limit point of $\{(u_j,v_j)\}_{j=1}^\infty \subset \mathcal{A} \cup \mathcal{T}$, then the result is obvious since

$$\limsup_{j\to\infty} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}(u_j,v_j) \geq \limsup_{j\to\infty} \frac{4}{r^4}(u_j,v_j) = \infty.$$

Thus, without loss of generality, we can assume that $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{R}$. Moreover, let us assume that (U,v) does not lie on $\mathcal{CH}_{i^+} \cup i^\square$. Fix $u_0 < U$ such that $(u_0,v) \in \mathcal{U} \cap \mathcal{Q}^+$. Since r extends continuously to zero on $\mathcal{S}^1_\Gamma \backslash \left(\mathcal{CH}_{i^+} \cup i^\square\right)$, it follows that

$$\lim_{u\to U} \int_{v-\delta}^{v+\delta} \lambda(u,v')\ \mathrm{d}v' = 0$$

in $\{u \geq u_0\} \cap \{|v'-v| \leq \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$, for all sufficiently small $\delta > 0$. Integration of (59) therefore gives

$$\int_{v-\delta}^{v+\delta} \lambda(u_0,v') \exp\left(-\int_{r(u,v')}^{r(u_0,v')} \frac{\frac{1}{r}\left(\frac{2m}{r}\right)}{1-\frac{2m}{r}}\ \mathrm{d}r\right)\ \mathrm{d}v' \leq \int_{v-\delta}^{v+\delta} \lambda(u,v')\ \mathrm{d}v' \to 0$$

as $u \to U$ in $\{|v'-v| \leq \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$. By compactness, there exists a constant $c > 0$ such that $\lambda(u_0,v') > c > 0$ for all $\{|v'-v| \leq \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$. Thus, it must be that

$$\lim_{u\to U} \int_{v-\delta}^{v+\delta} \exp\left(-\int_{r(u,v')}^{r(u_0,v')} \frac{\frac{1}{r}\left(\frac{2m}{r}\right)}{1-\frac{2m}{r}}\ \mathrm{d}r\right)\ \mathrm{d}v' = 0.$$

In particular, we obtain

$$\sup_{v-\delta\leq v'\leq v+\delta} \int_{r(u,v')}^{r(u_0,v')} \frac{\frac{1}{r}\left(\frac{2m}{r}\right)}{1-\frac{2m}{r}}\ \mathrm{d}r = \infty. \tag{64}$$

Let $\epsilon > 0$. Suppose for the moment that

$$\frac{2m}{r}(u,v') \leq r^\epsilon(u,v')$$

in $\{u \geq U - \hat{\epsilon}\} \cap \{|v'-v| \leq \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$ for some $\hat{\epsilon} > 0$. Then,

$$\sup_{v-\delta\leq v'\leq v+\delta} \int_{r(u,v')}^{r(u_0,v')} \frac{\frac{1}{r}\left(\frac{2m}{r}\right)}{1-\frac{2m}{r}}\ \mathrm{d}r \leq C(\epsilon,\hat{\epsilon}) < \infty,$$

contradicting (64). Thus, we conclude that there exists a sequence of points $(U_j,v_j) \in \{u \geq U - \hat{\epsilon}\} \cap \{|v'-v| \leq \delta\} \cap \mathcal{U} \cap \mathcal{Q}^+$ such that

$$\frac{2m}{r}(U_j,v_j) \geq r^\epsilon(U_j,v_j). \tag{65}$$

Letting $\delta \to 0$, we can construct a sequence $(U_{j_k},v_{j_k}) \to (U,v)$ for which (65) holds. The result follows from (63) since

$$\limsup_{j\to\infty} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}(U_{j_k},v_{j_k}) \geq \limsup_{j\to\infty} \frac{4}{r^{4-2\epsilon}}(U_{j_k},v_{j_k}) = \infty.$$

Lastly, we note that if (U,v) does lie on $\mathcal{CH}_{i^+} \cup i^\square$, then there is a sequence of points $(U,v_j) \to (U,v)$ on $\mathcal{S}^1_\Gamma \backslash (\mathcal{CH}_{i^+} \cup i^\square)$ for which the above argument applies. In particular, there exists a sequence of points (U_i,v_j) such that the Kretschmann scalar blows up as $U_i \to U$. Thus, we can construct a suitable subsequence of points (U_{i_k},v_{j_k}) for which the Kretschmann scalar blows up as $(U_{i_k},v_{j_k}) \to (U,v)$.

(b) As in Claim 1, given $p \in \mathcal{S}^1_\Gamma$, if there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{1+1}$ of p such that $\mathcal{U} \cap \mathcal{Q}^+ \subset \mathcal{A} \cup \mathcal{T}$, then the Kretschmann scalar will extend continuously to ∞ on $\mathcal{S}^1_\Gamma \cap \mathcal{U}$.

5.8.3 Claim 3

We deduce from Theorem 1.11 that if $\mathcal{BH} \neq \emptyset$, then $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ has a future boundary in $\mathcal{Q}^+$, which we call $\mathcal{H}^+$. In fact, for $\mathcal{I}^+ \subset \{v = V\}$, we have

$$\mathcal{H}^+ = \{u = U\} \cap \mathcal{Q}^+ \subset \mathcal{R} \cup \mathcal{A}$$

for some U. Fix $v_0 < V$ and consider the set $\mathcal{H}^+ \cap \{v \geq v_0\} = \{U\} \times [v_0, V)$. Without loss of generality, we can assume that $(U, v_0) \notin \Gamma$.

To prove the claim we will establish the contrapositive, namely: If $\mathcal{H}^+$ is not affine complete, i.e.

$$\int_{v_0}^{V} \Omega^2(U, v)\, \mathrm{d}v \leq C < \infty,$$

then $\mathcal{CH}_{i^+} \neq \emptyset$. In particular, we wish to show that, under the assume that $\mathcal{H}^+$ is not affine complete, there exists $u_* > U$ and $r_0 > 0$ such that $r \geq \frac{1}{2} r_0$ in

$$\mathcal{D} = ([U, u_*] \times [v_0, V]) \cap \mathcal{Q}^+.$$

We proceed with a bootstrap argument.

Let $r_0 = r(U, v_0)$ and define a region $\mathcal{D}^* \subset \mathcal{D}$ to be the set of $p = (u, v) \in \mathcal{D}$ such that

$$r(\tilde{u}, \tilde{v}) > \frac{1}{2} r_0 \tag{66}$$

for all $(\tilde{u}, \tilde{v}) \in (J^-(p) \backslash \{p\}) \cap \mathcal{D}$. Since $r \geq r_0$ along $\mathcal{H}^+$, it is clear that, by continuity, $\mathcal{D}^*$ is a non-empty open set (in the topology of $\mathcal{D}$). We will show that $\mathcal{D}^*$ is also closed (in the topology of $\mathcal{D}$). To see this, suppose $p \in \overline{\mathcal{D}^*}$ (in the topology of $\mathcal{D}$). By continuity, (66) holds in $J^-(p) \cap \mathcal{D}$, where, however, strict-inequality is replaced by non-strict inequality.

From (60), we note that

$$\partial_v(-\nu) = \frac{1}{2r^2}\left(m - \frac{2\pi\sigma^2}{r}\right)\Omega^2 \leq \frac{1}{2r^2} m \Omega^2. \tag{67}$$

Since $\mathcal{H}^+ \subset \mathcal{R} \cup \mathcal{A}$, it follows from (53) that

$$m(U, v') \leq \frac{1}{2} r(U, v').$$

Integration along $\{U\} \times [v_0, v]$ then gives

$$-\nu(U, v) \leq -\nu(U, v_0) + \int_{v_0}^{v} \frac{1}{2r^2} m \Omega^2(U, \overline{v})\, \mathrm{d}\overline{v} \leq C < \infty.$$

Let $u' \in [U, u]$. By monotonicity (21), we have that λ can change sign at most once along an outgoing null segment. Thus,

$$\int_{v_0}^{v} |\lambda(u', \overline{v})|\, \mathrm{d}\overline{v} \leq C < \infty. \tag{68}$$

Define the function κ by

$$\Omega^2 = -4\kappa\nu.$$

By monotonicity (20), $\partial_u \kappa \leq 0$. In particular,

$$\int_{v_0}^{v} \kappa(u', \overline{v})\, \mathrm{d}\overline{v} \leq -\frac{1}{4} \int_{v_0}^{v} \Omega^2 \nu(U, \overline{v})\, \mathrm{d}\overline{v} \leq C < \infty. \tag{69}$$

Since (67) gives

$$\partial_v \log(-\nu) = \frac{2\kappa}{r^2}\left(m - \frac{2\pi\sigma^2}{r}\right) \leq \frac{\kappa}{r}\left(\frac{2m}{r}\right), \tag{70}$$

let us consider the sets

$$\mathcal{V}_1 = \left\{\frac{2m}{r} \geq 2\right\} \cap \mathcal{D}^* \qquad \text{and} \qquad \mathcal{V}_2 = \left\{\frac{2m}{r} \leq 2\right\} \cap \mathcal{D}^*.$$

On the set $\mathcal{V}_1$, we note that

$$0 < \frac{\frac{2m}{r}}{1 - \frac{2m}{r}} \leq 2.$$

We then have that (68) and (69) give a uniform bound on the integral

$$\int_{v_0}^{v} \frac{\kappa}{r}\left(\frac{2m}{r}\right) \mathrm{d}\overline{v} \;=\; \int_{[v_0,v]\cap\mathcal{V}_1} \frac{\lambda}{r}\left(\frac{\frac{2m}{r}}{1-\frac{2m}{r}}\right) \mathrm{d}\overline{v} + \int_{[v_0,v]\cap\mathcal{V}_2} \frac{\kappa}{r}\left(\frac{2m}{r}\right) \mathrm{d}\overline{v},$$

where we have used our bootstrap assumption. In particular, integration of (70) gives the estimate

$$-\nu(u',v) \leq -C\nu(u',v_0) \leq C \sup_{U\leq u'\leq u_*} |\nu(u',v_0)| \leq C < \infty,$$

from which integration in u then gives

$$r(u,v) \geq r(U,v) - C(u-U) \geq r_0 - C(u-U).$$

For sufficiently small $u_* - U$ we obtain

$$r(u,v) \geq \frac{3}{4} r_0.$$

Thus, $(u,v) \in \mathcal{D}^*$ and we have that $\mathcal{D}^*$ is a closed set. Since $\mathcal{D}^*$ is connected, it follows that

$$\mathcal{D}^* = \mathcal{D}.$$

This establishes the claim.

5.8.4 Claim 4

Suppose $(\mathcal{M}, g_{\mu\nu})$ is future-extendible as a C^2-Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g_{\mu\nu}})$. Then, there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting $\mathcal{M}$ such that the closure of the projection of $\gamma|_{\mathcal{M}}$ to $\mathcal{Q}^+$ intersects the boundary $\mathcal{B}^+\backslash i^0$. Let $\mathcal{E}$ denote the set of all such boundary points on $\mathcal{B}^+\backslash i^0$ satisfying the above, i.e.

$$\mathcal{E} = \Big\{ p \in \mathcal{B}^+\backslash i^0 : \exists \text{ timelike } \gamma : I \to \widetilde{\mathcal{M}} \;\text{ with }\; \gamma(t^*) \in \partial\mathcal{M} \subset \widetilde{\mathcal{M}} \;\text{ and}$$
$$\gamma(t) \in \mathcal{M} \;\; \forall t < t^* \;\; \text{s.t.} \;\; \{p\} \cap \overline{\pi\left(\gamma|_{\mathcal{M}}\right)} \neq \emptyset \Big\}.$$

It is important to emphasize that extendibility is not formulated, *per se*, with respect to the quotient manifold $\mathcal{Q}^+$. It we were able, however, to assert that past set structure 'upstairs' is preserved 'downstairs', then criteria can be given on $\mathcal{Q}^+$ that will imply inextendibility of $\mathcal{M}$. These criteria on $\mathcal{Q}^+$ will be given by C^2-compatible scalar invariants, which we now introduce.

C^2-compatible scalar invariants

Let γ be as above and let $x \in \partial\mathcal{M} \subset \widetilde{\mathcal{M}}$ be the point through which γ exits the spacetime and consider $x' \in \gamma|_{\mathcal{M}}$ sufficiently close to x such that $J^+(x') \cap J^-(x) \subset \widetilde{\mathcal{M}}$ is compact.

We call Ξ a C^2-compatible scalar invariant on $\widetilde{\mathcal{M}}$ if Ξ remains uniformly bounded in the spacetime region $J^+(x') \cap J^-(x) \cap \mathcal{M}$ for x' sufficiently close to x.

Of interest to us are two particular C^2-compatible scalar invariants, namely: the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ and $g(X, X)$, where X is any Killing vector field, i.e. X satisfies

$$\nabla_\nu \nabla_\alpha X_\beta = R_{\mu\nu\alpha\beta} X^\mu.$$

That, indeed, X is at least everywhere continuous (in fact, C^1) see [31].

Extendibility criteria on $\mathcal{Q}^+$

Consider the spacetime region $P = I^-(\gamma|_{\mathcal{M}}) \subset \mathcal{M}$. Note that the set P satisfies $P = I^-(P)$, i.e. P is a past set in $\mathcal{M}$. If we can show that

$$\pi(P) = I^-(\pi(P)) \cap \mathcal{Q}^+, \tag{71}$$

that is, $\pi(P)$ is also a past set in $\mathcal{Q}^+$, then for $p \in \mathcal{E}$ it follows that

1. $I^-(p) \cap \mathcal{Q}^+ \subset \pi(P)$; and,
2. there exists a $p' \in \pi(\gamma|_{\mathcal{M}})$ such that every C^2-compatible scalar invariant Ξ is uniformly bounded in the region $J^-(p) \cap J^+(p') \cap \mathcal{Q}^+$.

In particular, once we have shown that past set structure is preserved, showing that there does not exist a timelike curve γ that exits the spacetime through $x \in \partial\mathcal{M}$ reduces to finding a C^2-compatible scalar invariant Ξ that is *not* uniformly bounded in $J^-(p) \cap J^+(p') \cap \mathcal{Q}^+$ *for every* $p' \in \pi(\gamma|_{\mathcal{M}})$.

Past set structure is preserved

It is easy to show that since the metric h on $\mathbb{S}^2$ is positive definite, timelike (resp. causal) vectors on $T\mathcal{M}_p$, for $p \in \mathcal{M}$, project to timelike (resp. causal) vectors on $T\mathcal{Q}^+_{\pi(p)}$. On the other hand, null vectors need not project to null vectors, for if $\pi_h : \mathcal{M} \to \mathbb{S}^2$ denotes the standard projection map, then a null vector $V \in T\mathcal{M}_p$ will map to a timelike vector $\pi(V) \in T\mathcal{Q}^+_{\pi(p)}$ unless $(\pi_h)_* V = 0$, in which case $\pi(V)$ is also null. Lastly, note that the horizontal lift of timelike (resp. causal) vectors in $T\mathcal{Q}^+_q$ are timelike (resp. causal) vectors in $T\mathcal{M}_p$, with $\pi(p) = q$.

Let $\gamma|_{\mathcal{M}} \subset \mathcal{M}$ be a timelike curve and consider the past set $P = I^-(\gamma|_{\mathcal{M}}) \subset \mathcal{M}$. We claim that

$$\pi(P) = I^-(\pi(\gamma|_{\mathcal{M}})) \cap \mathcal{Q}^+. \tag{72}$$

Let $q \in \pi(P)$. Then, there exists $p \in P$ such that $\pi(p) = q$. Since P is a past set, it follows that for some $p' \in \gamma|_{\mathcal{M}}$, there exists a timelike curve $\gamma_p : I \to \mathcal{M}$ such that $\gamma_p(0) = p$ and $\gamma_p(1) = p'$. We then define a timelike vector field $X \in T\mathcal{M}$ along γ_p such that

$$X(t) = \dot{\gamma}_p(t).$$

This vector field then projects to a timelike vector field $\pi(X) \in T\mathcal{Q}^+$ along $\pi(\gamma_p)$. In particular, there exists a future-directed timelike curve that connects q and $\pi(p') \in \pi(\gamma|_{\mathcal{M}})$, i.e. $q \in I^-(\pi(\gamma|_{\mathcal{M}})) \cap \mathcal{Q}^+$.

Now suppose that $q \in I^-(\pi(\gamma|_{\mathcal{M}})) \cap \mathcal{Q}^+$. Then, by definition, there exists $q' \in \pi(\gamma|_{\mathcal{M}})$ and a timelike curve $\gamma_q : I \to \mathcal{Q}^+$ such that $\gamma_q(0) = q$ and $\gamma_q(1) = q'$. We define a timelike vector field $Y \in T\mathcal{Q}^+$ along γ_q such that

$$Y(t) = \dot{\gamma}_q(t).$$

The curve γ_q has a unique horizontal lift $\hat{\gamma}_q$ in $\mathcal{M}$ through $p' \in \gamma|_{\mathcal{M}}$ such that $\pi(\hat{\gamma}_p) = \gamma_q$ and $\pi(p') = q'$. Since the horizontal lift of a timelike vector is also timelike, it follows that there exists a timelike vector field $\hat{Y} \in T\mathcal{M}$ along $\hat{\gamma}_q$ such that $\pi(\hat{Y}) = Y$. We then follow along the lifted timelike curve from p' to the point $p \in P$ such that $\pi(p) = q$. In particular, $q \in \pi(P) \cap \mathcal{Q}^+$.

This establishes (72) and we obtain

$$I^-(\pi(P)) \cap \mathcal{Q}^+ = I^-(\pi(\gamma|_{\mathcal{M}})) \cap \mathcal{Q}^+ = \pi(P),$$

which yields (71).

Characterization of the set $\mathcal{E}$

Suppose $\mathcal{E} \cap (\mathcal{S}_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}) \neq \emptyset$ and let $p \in \mathcal{E} \cap (\mathcal{S}_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+})$. By definition, there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting $\mathcal{M}$ such that $\{p\} \cap \overline{\pi(\gamma|_{\mathcal{M}})} \neq \emptyset$. For all $p' \in \pi(\gamma|_{\mathcal{M}})$, Claims 1 and 2 assert that the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is not uniformly bounded in $J^-(p) \cap J^+(p') \cap \mathcal{Q}^+$. In view of (2) above, this contradicts the assumption that $p \in \mathcal{E}$. We conclude therefore that $\mathcal{E} \cap (\mathcal{S}_\Gamma \cup \mathcal{S} \cup \mathcal{S}_{i^+}) = \emptyset$.

Suppose $\mathcal{E} \cap \left(i^\square \cup \mathcal{I}^+\right) \neq \emptyset$ and let $p \in \mathcal{E} \cap \left(i^\square \cup \mathcal{I}^+\right)$. By definition, there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting $\mathcal{M}$ such that $\{p\} \cap \overline{\pi(\gamma|_{\mathcal{M}})} \neq \emptyset$. We recall the fact that in spherical symmetry there are three Killing vector fields X_1, X_2 and X_3 such that

$$g(X_1, X_1) + g(X_2, X_2) + g(X_3, X_3) = 2r^2.$$

Since r is unbounded in $J^+(p'') \cap J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ for all $p'' \in \pi(\gamma|_{\mathcal{M}})$, it follows that, without loss of generality, $g(X_1, X_1)$ is unbounded as well. In view of (2) above, this contradicts the assumption that $p \in \mathcal{E}$. We conclude therefore that $\mathcal{E} \cap \left(i^\square \cup \mathcal{I}^+\right) = \emptyset$.

We now claim that if $b_\Gamma \in \mathcal{E}$, then $\mathcal{E} \cap (\mathcal{CH}_\Gamma \cup \mathcal{CH}_{i^+}) \neq \emptyset$.

Let $(U, V) = b_\Gamma \in \mathcal{E}$. By definition, there exists a timelike curve $\gamma \subset \widetilde{\mathcal{M}}$ exiting $\mathcal{M}$ such that $b_\Gamma \cap \overline{\pi(\gamma|_{\mathcal{M}})} \neq \emptyset$. Given a sequence $(U_j, V) \to (U, V)$, with $U_j \leq U$, we note that

$$\frac{2m}{r}(U_j, V) \not\to 1,$$

as $U_j \to U$. For, if $\frac{2m}{r}(U_j, V) \to 1$, then $\limsup_{j\to\infty} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}(U_j, V) = \infty$, contradicting the assumption that $b_\Gamma \in \mathcal{E}$. In particular, we conclude that $\mathcal{A} \cap J^-(b_\Gamma)$ does not have a limit point on b_Γ. As a result, we may assume, without loss of generality, that for fixed $v' < V$ and $u' < U$, the region

$$\mathcal{D} = ([u', U] \times [v', V]) \cap \mathcal{Q}^+$$

satisfies $\mathcal{D} \cap (\mathcal{A} \cup \mathcal{T}) = \emptyset$. Moreover, without loss of generality, we can assume that b_Γ does not coincide with $\mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^\square$. (If $b_\Gamma \in \mathcal{S}_{i^+} \cup i^\square$, then $b_\Gamma \notin \mathcal{E}$ as argued above.[52] If $b_\Gamma \in \mathcal{CH}_{i^+}$, then there is nothing to show.)

It follows that on the compact set $(\{u'\} \times [v', V] \cup [u', U] \times \{v'\}) \cap \mathcal{Q}^+$, there exists a constant $c > 0$ such that $1 - \frac{2m}{r} > c$. Thus, if we re-normalize the co-ordinates on $\mathcal{D}$ such that

$$\frac{-\nu}{1 - \frac{2m}{r}}(u, V) = 1 \qquad \text{and} \qquad \frac{\lambda}{1 - \frac{2m}{r}}(u', v) = 1,$$

[52] In fact, we have already implicitly ruled out the possibility that b_Γ coincides with $\mathcal{S}_{i^+}$, as $\mathcal{D} \cap (\mathcal{A} \cup \mathcal{T}) = \emptyset$.

then these new co-ordinates u and v will have finite range and $b_\Gamma = (U, V)$. By monotonicity (57) and (62), it follows that

$$\frac{-\nu}{1-\frac{2m}{r}}(u,v) \le 1 \qquad \text{and} \qquad \frac{\lambda}{1-\frac{2m}{r}}(u,v) \le 1,$$

for all $(u,v) \in \mathcal{D}$. In particular, we have that Ω^2 is uniformly bounded in $\mathcal{D}$, for (53) yields

$$\frac{1}{4}\Omega^2 = \frac{-\lambda\nu}{1-\frac{2m}{r}} = \frac{\lambda}{1-\frac{2m}{r}}\frac{-\nu}{1-\frac{2m}{r}}\left(1-\frac{2m}{r}\right) \le 1. \tag{73}$$

We now note that since the extension $\widetilde{\mathcal{M}}$ is a regular manifold, if γ exits through $x \in \partial\mathcal{M} \subset \widetilde{\mathcal{M}}$, then there must be an open neighborhood $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{M}}$ of x through which a suitable family of timelike curves (to be defined later) also exits into the extension. We can therefore require, without loss of generality, that $\{p \in \mathcal{M} : g \cdot p = p,\ \forall g \in SO(3)\} \cap \widetilde{\mathcal{U}} = \emptyset$ for a sufficiently small neighborhood $\widetilde{\mathcal{U}}$, i.e. γ does not intersect the the set of fixed points of the $SO(3)$-action (the center of symmetry Γ 'downstairs').

Given $\vec{s} = (s_1, s_2, s_3) \in B^3(0) \subset \mathbb{R}^3$, we consider a 3-parameter family of curves $\gamma_{\vec{s}}$ given by

$$\gamma_{\vec{s}} : t \in (-\epsilon, \epsilon) \mapsto (t, \vec{s})$$

with $\gamma_{(0,0,0)} = \gamma$ such that $\gamma_{\vec{s}}(0) \in \mathcal{M} \cap \widetilde{\mathcal{U}}$ and $\gamma_{\vec{s}}(\frac{1}{2}\epsilon) \notin \mathcal{M}$. Let us choose $\widetilde{\mathcal{U}}$ suitably small such that the mapping

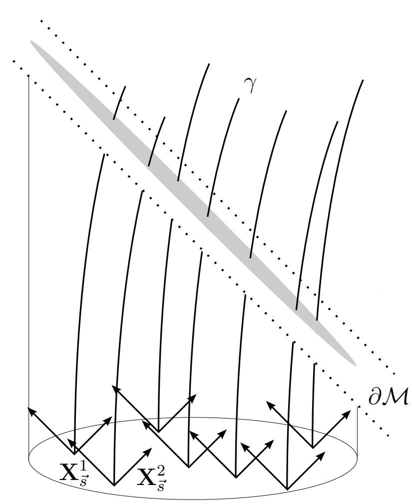
$$\Phi_0 : (-\epsilon, \epsilon) \times B^3(0) \to \widetilde{\mathcal{M}}$$

is a diffeomorphism on its image $\Phi_0[(t, \vec{s})]$. For each of these curves, let us define a pair of orthogonal null vectors $\mathbf{X}^1_{\vec{s}}$ and $\mathbf{X}^2_{\vec{s}}$ tangent to $\gamma_{\vec{s}}(0)$ whose projection to $\mathcal{Q}^+$ is given by

$$\pi(\mathbf{X}^1_{\vec{s}})(\pi(\gamma_{\vec{s}}(0))) = a(\vec{s})\frac{\partial}{\partial u} \qquad \text{and} \qquad \pi(\mathbf{X}^2_{\vec{s}})(\pi(\gamma_{\vec{s}}(0))) = b(\vec{s})\frac{\partial}{\partial v},$$

for smooth real-valued functions $a(\vec{s}) = a(\pi(\gamma_{\vec{s}}(0)))$ and $b(\vec{s}) = b(\pi(\gamma_{\vec{s}}(0)))$ normalized such that

$$2 = -g(\mathbf{X}^1_{\vec{s}}, \mathbf{X}^2_{\vec{s}}).$$



Now, parallel transport the vectors $\mathbf{X}^1_{\vec{s}}$ and $\mathbf{X}^2_{\vec{s}}$ along each respective curve $\gamma_{\vec{s}}$. This defines two smooth vector fields $\mathbf{X}^1$ and $\mathbf{X}^2$ on $\Phi_0[(t, \vec{s})] \cap \widetilde{\mathcal{U}} \cap \mathcal{M}$. Because parallel transport preserves nullity, we note that

$$\mathbf{X}^1 = a\frac{\partial}{\partial u} \qquad \text{and} \qquad \mathbf{X}^2 = b\frac{\partial}{\partial v}$$

for smooth real-valued functions a and b defined on $\Phi_0[(t,\vec{s})]\cap\widetilde{\mathcal{U}}\cap\mathcal{M}$. Moreover, because parallel transport preserves the inner product of vectors it follows that

$$2=-g(\mathbf{X}^1,\mathbf{X}^2)=ab\Omega^2.$$

Since we have shown that Ω^2 is bounded (73), we must have that there exists a constant $\xi>0$ such that *either* $a>\xi$ or $b>\xi$.

Suppose $a>\xi$. Let $(s_1,s_2,0)\in B^2(0)\subset\mathbb{R}^2$ and consider a mapping

$$\Phi_1:(-\epsilon,\epsilon)\times B^2(0)\times(-\delta,\delta)\to\widetilde{\mathcal{M}}$$

defined by following the point $\gamma_{(s_1,s_2,0)}(t)$ along the (future-directed) integral curves of $\mathbf{X}^1$ for time $0\le\tau\le\frac{1}{2}\delta$. For δ suitably small, Φ_1 is a diffeomorphism onto its image $\Phi_1[(t,s_1,s_2,\tau)]$. In particular, since $\mathbf{X}^1$ is a smooth vector field, we have a tubular neighborhood $\widetilde{\mathcal{U}_{\delta/2}}=\Phi_1[(t,s_1,s_2,\tau)]\subset\Phi_0[(t,\vec{s})]\cap\widetilde{\mathcal{U}}$ for sufficiently small δ. Define $t^*(\vec{s})\in(0,\frac{1}{2}\epsilon]$ as the time t at which $\gamma_{\vec{s}}$ exits $\mathcal{M}$ and put $t^*=\inf t^*(\vec{s})>0$. Let $\widetilde{\mathcal{U}_{\delta/2}}(t^*)=\widetilde{\mathcal{U}_{\delta/2}}\cap\mathcal{M}$ denote the image of Φ_1 with $t<t^*$. It then follows that in this neighborhood

$$u\left(\Phi\left(t,s_1,s_2,\frac{1}{2}\delta\right)\right)-u\left(\Phi\left(t,s_1,s_2,0\right)\right)\ge\frac{1}{2}\delta\xi>0. \tag{74}$$

We now claim that $\pi(\widetilde{\mathcal{U}_{\delta/2}}(t^*))\not\subset\mathcal{D}$. Suppose, on the contrary, that $\pi(\widetilde{\mathcal{U}_{\delta/2}}(t^*))\subset\mathcal{D}$. Since

$$\pi\left(\widetilde{\mathcal{U}_{\delta/2}}(t^*)\right)\cap J^+(\pi\left(\gamma(0)\right))\neq\emptyset,$$

let p' be an element of this set. We note that, however, the u-dimension of $J^+(p')\cap\mathcal{D}$ tends to zero as $p'\to b_\Gamma$, i.e. for $p'',p'''\in J^+(p')\cap\mathcal{D}$ we have

$$\lim_{p'\to b_\Gamma}\sup_{J^+(p')\cap\mathcal{D}}|u(p'')-u(p''')|=0.$$

This then contradicts (74). We therefore conclude that $\pi(\widetilde{\mathcal{U}_{\delta/2}}(t^*))\not\subset\mathcal{D}$. Since we have shown that

$$\mathcal{E}\cap(\mathcal{S}_\Gamma\cup\mathcal{S}\cup\mathcal{S}_{i^+}\cup i^{\square}\cup\mathcal{I}^+)=\emptyset,$$

it must be the case that

$$\mathcal{E}\cap(\mathcal{CH}_\Gamma\cup\mathcal{CH}_{i^+})\neq\emptyset. \tag{75}$$

Suppose, on the other hand, that $b>\xi$. We consider a mapping

$$\Phi_2:(-\epsilon,\epsilon)\times B^2(0)\times(-\delta,\delta)\to\widetilde{\mathcal{M}}$$

defined by following the point $\gamma_{(s_1,s_2,0)}(t)$ along the (future-directed) integral curves of $\mathbf{X}^2$ for time $0\le\tau\le\frac{1}{2}\delta$. For δ suitably small, Φ_2 is a diffeomorphism onto its image $\Phi_2[(t,s_1,s_2,\tau)]$. In particular, since $\mathbf{X}^2$ is a smooth vector field, we have a tubular neighborhood $\widetilde{\mathcal{U}_{\delta/2}}=\Phi_2[(t,s_1,s_2,\tau)]\subset\Phi_0[(t,\vec{s})]\cap\widetilde{\mathcal{U}}$ for sufficiently small δ. Similarly, we define $\widetilde{\mathcal{U}_{\delta/2}}(t^*)$. Either a timelike curve in the tubular neighborhood $\widetilde{\mathcal{U}_{\delta/2}}(t^*)$ will exit $J^-(x)\cap\mathcal{M}$ (recall $x\in\partial\mathcal{M}$ is the point at which γ exits $\mathcal{M}$) so that (75) holds, in which case there is nothing to show, or every timelike curve in the neighborhood will remain in $J^-(x)\cap\mathcal{M}$, in which case

$$v\left(\Phi\left(t,s_1,s_2,\frac{1}{2}\delta\right)\right)-v\left(\Phi\left(t,s_1,s_2,0\right)\right)\ge\frac{1}{2}\delta\xi>0.$$

We argue as before, noting that for a given $p'\in\pi(\widetilde{\mathcal{U}_{\delta/2}}(t^*))\cap J^+(\pi\left(\gamma(0)\right))$ the v-dimension of $J^+(p')\cap\mathcal{D}$ tends to zero as $p'\to b_\Gamma$. $\square$

6 Proof of Theorems 1.12 and 1.13: global structure of strongly and weakly tame Einstein-matter systems

We note that only in §4, in which we established the generalized extension principle for the Einstein-Maxwell-Klein-Gordon system, did we exploit model-specific structure. Elsewhere, assertions were proven with the aid of simply the dominant (or the weaker null energy) condition. Accordingly, since both Theorems 1.12 and 1.13 presume that a suitable extension principle hold, we may reproduce the proof of either Theorem exactly as in §5.

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